TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
https://doi.org/10.1090/tran/7798
Article electronically published on June 10, 2019

REGULAR EXTENSIONS AND ALGEBRAIC RELATIONS BETWEEN VALUES OF MAHLER FUNCTIONS IN POSITIVE CHARACTERISTIC

GWLADYS FERNANDES

Abstract. Let $\mathbb{K}$ be a function field of characteristic $p > 0$. We have recently established the analogue of a theorem of Ku. Nishioka for linear Mahler systems defined over $\mathbb{K}(z)$. This paper is dedicated to proving the following refinement of this theorem. Let $f_1(z), \ldots, f_n(z)$ be d-Mahler functions such that $\overline{\mathbb{K}}(z)\,(f_1(z), \ldots, f_n(z))$ is a regular extension over $\overline{\mathbb{K}}(z)$. Then, every homogeneous algebraic relation over $\overline{\mathbb{K}}$ between their values at a regular algebraic point arises as the specialization of a homogeneous algebraic relation over $\overline{\mathbb{K}}(z)$ between these functions themselves. If $\mathbb{K}$ is replaced by a number field, this result is due to B. Adamczewski and C. Faverjon as a consequence of a theorem of P. Philippon. The main difference is that in characteristic zero, every d-Mahler extension is regular, whereas in characteristic p, non-regular d-Mahler extensions do exist. Furthermore, we prove that the regularity of the field extension $\overline{\mathbb{K}}(z)\,(f_1(z), \ldots, f_n(z))$ is also necessary for our refinement to hold. Besides, we show that when $p \nmid d$, d-Mahler extensions over $\overline{\mathbb{K}}(z)$ are always regular. Finally, we describe some consequences of our main result concerning the transcendence of values of d-Mahler functions at algebraic points.

1. Introduction

Let $\mathbb{K}$ be a field and let $d \geq 2$ be an integer. We say that a power series $f(z) \in \mathbb{K}[[z]]$ is a d-Mahler function over $\mathbb{K}(z)$ if there exist polynomials $P_0(z), \ldots, P_n(z) \in \mathbb{K}[z]$, $P_n(z) \not\equiv 0$, such that

$$P_0(z)f(z) + P_1(z)f(z^d) + \cdots + P_n(z)f(z^{d^n}) = 0. \tag{1.1}$$

The minimal integer n satisfying the previous equation is called the order of $f(z)$. We say that the column vector whose coordinates are the power series $f_1(z), \ldots, f_n(z) \in \mathbb{K}[[z]]$ satisfies a d-Mahler system if there exists a matrix $A(z) \in \mathrm{GL}_n(\mathbb{K}(z))$ such that

$$\begin{pmatrix} f_1(z^d) \\ \vdots \\ f_n(z^d) \end{pmatrix} = A(z) \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}. \tag{1.2}$$

Received by the editors August 21, 2018, and, in revised form, November 22, 2018.

2010 *Mathematics Subject Classification.* Primary 11J81, 11J72, 11J85.

Key words and phrases. Mahler's method, algebraic independence, linear independence, transcendence, regular extensions, function fields.

This project received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No. 48132.

Any d-Mahler function is a coordinate of a vector solution of the d-Mahler system associated with the companion matrix of (1.1). Reciprocally, every coordinate of a vector solution of a d-Mahler system is a d-Mahler function. We say that a number $\alpha \in \mathbb{K}$ is regular with respect to system (1.2) if for all integers $k \geq 0$, the number α^{d^k} is neither a pole of the matrix $A(z)$ nor a pole of the matrix $A^{-1}(z)$. In this paper, we are dealing with the case where $\mathbb{K}$ is a function field of positive characteristic. Let us introduce the associated framework. We start with a prime number p and we let $q = p^r$ denote a power of p. Then, we let $A = \mathbb{F}_q[T]$ denote the ring of polynomials in T, with coefficients in the finite field $\mathbb{F}_q$, and we let $K = \mathbb{F}_q(T)$ denote the fraction field of A. We define the $\frac{1}{T}$-adic absolute value on K by $\left|\frac{P(T)}{Q(T)}\right| = q^{\deg_T(P)-\deg_T(Q)}$. We recall that the completion of K with respect to $|.|$ is the field $\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$ of Laurent power series expansions over $\mathbb{F}_q$ and that the completion C of the algebraic closure of $\mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$ with respect to the unique extension of $|.|$ is a complete and algebraically closed field. Finally, as announced, we let $\mathbb{K}$ denote a function field, that is, a finite extension of K. We let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$, embedded in C.

Let $\mathbb{K}\{z\}$ denote the set of functions which admit a convergent power series expansion on a domain containing the origin, with coefficients in $\mathbb{K}$. Let $\boldsymbol{k}$ be a field and let $\mathcal{F}$ be a finite family of elements of a $\boldsymbol{k}$-algebra. We let $\operatorname{trdeg}_{\boldsymbol{k}}\{\mathcal{F}\}$ denote the transcendence degree of $\mathcal{F}$ over $\boldsymbol{k}$, that is, the maximal number of elements of $\mathcal{F}$ that are algebraically independent over $\boldsymbol{k}$. In [10], the author has proved the following result. This is the analogue for function fields of characteristic p of a classical result due to Ku. Nishioka [19] when $\mathbb{K}$ is a number field.

Theorem 1.1 (F). *Let $n \geq 1$, $d \geq 2$ be two integers and let $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be functions satisfying d-Mahler system* (1.2). *Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, be a regular number with respect to system* (1.2). *Then*

$$\operatorname{trdeg}_{\overline{\mathbb{K}}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \operatorname{trdeg}_{\overline{\mathbb{K}}(z)}\{f_1(z), \ldots, f_n(z)\}. \tag{1.3}$$

In general, little is known about the algebraic relations between the functions $f_1(z), \ldots, f_n(z)$ over $\overline{\mathbb{K}}(z)$. This makes a priori difficult the question whether $f(\alpha)$ is transcendental or not over $\overline{\mathbb{K}}$. However, it is easier to study linear relations between the functions $f_1(z), \ldots, f_n(z)$ over $\overline{\mathbb{K}}(z)$. For example, when $\mathbb{K}$ is a number field, a basis of the set of linear relations over $\overline{\mathbb{Q}}(z)$ between the Mahler functions $f_1(z), \ldots, f_n(z)$ can be explicitly computed [1,2]. The arguments used by B. Adamczewski and C. Faverjon to obtain this result belong to linear algebra and might fit for function fields. This could be a further perspective for study. For these reasons, we are interested in refining Theorem 1.1. The following definition is needed.

Definition 1.2. Let $\boldsymbol{k}$ be a field. We say that a finitely generated field extension $\mathcal{E} = \boldsymbol{k}(u_1, \ldots, u_n)$ of $\boldsymbol{k}$ is regular over $\boldsymbol{k}$ if the following two conditions are satisfied.

1. $\mathcal{E}$ is separable over $\boldsymbol{k}$. That is, there exists a transcendence basis $\mathcal{F}$ of $\mathcal{E}$ over $\boldsymbol{k}$ such that $\mathcal{E}$ is a separable algebraic extension of $\boldsymbol{k}(\mathcal{F})$ (see [9, Appendix A1.2] and also [17]).
2. Every element of $\mathcal{E}$ that is algebraic over $\boldsymbol{k}$ belongs to $\boldsymbol{k}$.

With this definition, our main result is the following.

Theorem 1.3. *We continue with the assumptions of Theorem* 1.1. *Let us assume further that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$.*

Then, for every polynomial $P(X_1,\ldots,X_n) \in \overline{\mathbb{K}}[X_1,\ldots,X_n]$ *homogeneous in* $X_1,\ldots,X_n$ *such that*

$$P(f_1(\alpha),\ldots,f_n(\alpha)) = 0,$$

there exists a polynomial $Q(z,X_1,\ldots,X_n) \in \overline{\mathbb{K}}[z][X_1,\ldots,X_n]$ *homogeneous in* $X_1,\ldots,X_n$ *such that*

$$Q(z,f_1(z),\ldots,f_n(z)) = 0$$

and

$$Q(\alpha,X_1,\ldots,X_n) = P(X_1,\ldots,X_n).$$

Let us note that any inhomogeneous algebraic relation

$$P(f_1(\alpha),\ldots,f_n(\alpha)) = 0$$

can be turned into a homogeneous algebraic relation between the values at α of the functions $f_i(z)$ and the additional function 1.

As announced, Theorem 1.3 allows us to deal with linear independence over $\overline{\mathbb{K}}$ between values of Mahler functions.

Corollary 1.4. *We continue with the assumptions of Theorem* 1.3*. If the functions* $f_1(z),\ldots,f_n(z)$ *are linearly independent over* $\overline{\mathbb{K}}(z)$*, then the numbers* $f_1(\alpha),\ldots,f_n(\alpha)$ *are linearly independent over* $\overline{\mathbb{K}}$*.*

Given $f(z)$ a Mahler function, one of the main goals of Mahler's method is to decide whether $f(\alpha)$ is transcendental or not over $\overline{\mathbb{K}}$. Corollary 1.4 applied with the functions $1, f(z)$ shows the contribution of Theorem 1.3 in understanding the nature of $f(\alpha)$ when α is regular. Corollary 1.5 below states that this contribution even extends to the case of non-regular numbers α. Let us start with a single transcendental d-Mahler function $f(z)$. Then, there exist an integer $m \geq 1$ and coprime polynomials $P_{-1}(z),\ldots,P_m(z) \in \mathbb{K}[z]$, $P_m(z) \not\equiv 0$, such that

$$P_{-1}(z) + P_0(z)f(z) + P_1(z)f(z^d) + \cdots + P_m(z)f(z^{d^m}) = 0. \tag{1.4}$$

If m is minimal, we call (1.4) the minimal inhomogeneous equation of $f(z)$ over $\mathbb{K}(z)$. We can associate with this equation the d-Mahler system

$$\begin{pmatrix} 1 \\ f(z^d) \\ \vdots \\ f(z^{d^m}) \end{pmatrix} = A(z) \begin{pmatrix} 1 \\ f(z) \\ \vdots \\ f(z^{d^{m-1}}) \end{pmatrix}, \tag{1.5}$$

where $A(z) \in \mathrm{GL}_{m+1}(\mathbb{K}(z))$ is the companion matrix of (1.4). Let σ_d denote the endomorphism of $\mathbb{K}\{z\}$ defined by $\sigma_d g(z) = g(z^d)$. Then, we set

$$\overline{\mathbb{K}}(z)(g(z))_{\sigma_d} = \overline{\mathbb{K}}(z)\left(\{\sigma_d^i g(z)\}_{i\geq 0}\right).$$

Now, let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, be a regular number for system (1.5). The only thing we know a priori is that

$$\mathrm{trdeg}_{\overline{\mathbb{K}}(z)}\{1, f(z),\ldots,f(z^{d^{m-1}})\} \geq 1.$$

Therefore, Theorem 1.1 only gives

$$\mathrm{trdeg}_{\overline{\mathbb{K}}}\{1, f(\alpha),\ldots,f(\alpha^{d^{m-1}})\} \geq 1.$$

That is, there exists at least one transcendental number among $f(\alpha), \dots, f(\alpha^{d^{m-1}})$. But we cannot conclude that $f(\alpha)$ is transcendental. Our contribution to this problem is the following result.

Corollary 1.5. *Let $f(z) \in \mathbb{K}\{z\}$ be a d-Mahler transcendental function over $\mathbb{K}(z)$. Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ such that α is in the disc of convergence of $f(z)$. Let us assume that the extension $\overline{\mathbb{K}}(z)(f(z))_{\sigma_d}$ is regular over $\overline{\mathbb{K}}(z)$.*

Then, we have the following.

1. *The number $f(\alpha)$ is either transcendental or in $\mathbb{K}(\alpha)$.*
2. *If α is a regular number with respect to d-Mahler system* (1.5) *satisfied by $f(z)$ (that is, $P_0(\alpha^{d^k})P_m(\alpha^{d^k}) \neq 0$ for every integer $k \geq 0$), then $f(\alpha)$ is transcendental over $\overline{\mathbb{K}}$.*

Such results were first established in the setting of linear differential equations over $\overline{\mathbb{Q}}(z)$, especially for E-functions. Theorem 1.1 is the analogue of Siegel–Shidlovskii's theorem [26]. Theorem 1.3 is the analogue of a theorem of F. Beukers [6]. F. Beukers's proof uses Galois theory and results from Y. André. Moreover, Y. André proved [5] that the theorem of F. Beukers can be deduced from Siegel-Shidlovskii's theorem using a new method involving the theory of affine quasi-homogeneous varieties. Finally, the analogue of Corollary 1.5 for E-functions is stated in [11] (see also [4]). Getting back to Mahler functions, Theorem 1.3 is the analogue for function fields of a theorem of B. Adamczewski and C. Faverjon [2], obtained as a consequence of a result of P. Philippon [21]. The analogues of Corollaries 1.4 and 1.5 for number fields are proved in [2].

Besides, if $f_1(z), \dots, f_n(z)$ are either E-functions or Mahler functions over $\overline{\mathbb{Q}}(z)$, the extension $\overline{\mathbb{Q}}(z)\,(f_1(z), \dots, f_n(z))$ is always regular over $\overline{\mathbb{Q}}(z)$. This is straightforward for E-functions for they are analytic on the whole complex plane. For Mahler functions, this can be deduced [2, 21] from the fact that a Mahler function with coefficients in $\overline{\mathbb{Q}}$ is either rational or transcendental [20, Theorem 5.1.7]). But when $\mathbb{K}$ is a function field of characteristic p, such a dichotomy does not hold anymore and there do exist non-regular Mahler extensions. Let us provide a trivial example based on the following p-Mahler system:

$$\begin{pmatrix} f_1(z^p) \\ f_2(z^p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}.$$

A solution to this system is given by

$$f_1(z) = 1, f_2(z) = \sum_{n=0}^{+\infty} z^{p^n}.$$

Furthermore, $f_2(z)$ is algebraic because $f_2(z)^p = f_2(z^p) = f_2(z) - z$. On the other hand, the sequence of coefficients of $f_2(z)$ is not eventually periodic. Therefore, $f_2(z)$ is not rational. It follows that the extension $\mathcal{E} = \overline{\mathbb{K}}(z)(f_1(z), f_2(z))$ is not regular over $\overline{\mathbb{K}}(z)$. Now, let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$, and $\lambda = f_2(\alpha) \in \overline{\mathbb{K}}$. Then, $\lambda f_1(\alpha) - f_2(\alpha) = 0$ is a non-trivial linear relation between $f_1(\alpha)$ and $f_2(\alpha)$ over $\overline{\mathbb{K}}$. However, there is no non-trivial linear relation between the function $f_1(z)$ and $f_2(z)$ over $\overline{\mathbb{K}}(z)$, because $f_2(z)$ is not rational. Hence, the conclusion of Theorem 1.3 does not hold in this case. In Theorem 1.6, we state that this example reflects a general behavior. That is, the conclusion of Theorem 1.3 is never satisfied when the extension $\overline{\mathbb{K}}(z)\,(f_1(z), \dots, f_n(z))$ is not regular over $\overline{\mathbb{K}}(z)$.

Let us first introduce some definitions and notation. Let $\boldsymbol{k}$ be a valued field and let $\boldsymbol{k}_c$ be its completion. Note that its valuation extends uniquely to $\overline{\boldsymbol{k}_c}$ [25, II.2, Corollary 2]. We let $\tilde{\boldsymbol{k}}$ denote the completion of $\overline{\boldsymbol{k}_c}$ with respect to this valuation. Then, $\tilde{\boldsymbol{k}}$ is complete and algebraically closed. Now, let $\alpha \in \tilde{\boldsymbol{k}}$. We say that a function is analytic at α if it admits a convergent power series expansion in a connected open neighbourhood of α, with coefficients in $\tilde{\boldsymbol{k}}$. If $\mathcal{U} \subseteq \tilde{\boldsymbol{k}}$ is a domain, we say that a function is analytic on $\mathcal{U}$ if it is analytic at each point of $\mathcal{U}$. If the power series expansion of $f(z)$ at $\alpha \in \mathcal{U}$ has coefficients in a subfield L of $\tilde{\boldsymbol{k}}$, we say that $f(z)$ is analytic at α over L and we let $L\{z-\alpha\}$ denote the set of all such functions. Now, let $f_1(z), \ldots, f_n(z) \in \boldsymbol{k}\{z\}$. We set

$$\mathfrak{p} = \{Q(z, X_1, \ldots, X_n) \in \overline{\boldsymbol{k}}(z)[X_1, \ldots, X_n], Q(z, f_1(z), \ldots, f_n(z)) = 0\}.$$

If the functions $f_1(z), \ldots, f_n(z)$ are analytic at $\alpha \in \overline{\boldsymbol{k}}$, we set

$$\mathfrak{p}_\alpha = \{P(X_1, \ldots, X_n) \in \overline{\boldsymbol{k}}[X_1, \ldots, X_n], P(f_1(\alpha), \ldots, f_n(\alpha)) = 0\}.$$

Let R be a ring. If $\mathfrak{q}$ is an ideal of $A = R[X_1, \ldots, X_n]$, we write $\tilde{\mathfrak{q}}$ to refer to the homogenized ideal of $\mathfrak{q}$. It is the ideal of $A' = R[X_0, X_1, \ldots, X_n]$ generated by all the homogeneous polynomials $Q(X_0, \ldots, X_n) \in A'$ for which there exists a polynomial $P(X_1, \ldots, X_n) \in \mathfrak{q}$ such that $Q(1, X_1, \ldots, X_n) = P(X_1, \ldots, X_n)$. Finally, let $\mathrm{ev}_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n])$ denote the homogeneous ideal over $\overline{\boldsymbol{k}}[X_0, \ldots, X_n]$ constructed by evaluating the ideal $\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]$ at $z = \alpha$. With these definitions, the conclusion of Theorem 1.3 implies the following assertion:

$$\mathrm{ev}_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]) = \tilde{\mathfrak{p}}_\alpha.$$

Now, we can state the announced result.

Theorem 1.6. *Let $\boldsymbol{k}$ be a valued field. We assume that $f_1(z), \ldots, f_n(z) \in \overline{\boldsymbol{k}}\{z\}$ are analytic functions on a domain $\mathcal{U} \subseteq \tilde{\boldsymbol{k}}$ which contains the origin. Let $\alpha \in \mathcal{U} \cap \overline{\boldsymbol{k}}$. Let us assume further that the extension $\overline{\boldsymbol{k}}(z)(f_1(z), \ldots, f_n(z))$ is not regular over $\overline{\boldsymbol{k}}(z)$. Then we have*

$$ev_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]) \subsetneq \tilde{\mathfrak{p}}_\alpha.$$

In particular, the conclusion of Theorem 1.3 *does not hold.*

Besides, let $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be d-Mahler functions over $\mathbb{K}(z)$. Then, we say that the field extension $\overline{\mathbb{K}}(z)\,(f_1(z), \ldots, f_n(z))$ is a d-Mahler extension over $\overline{\mathbb{K}}(z)$ or, for short, d-Mahler. Now, if $p \nmid d$, we show that d-Mahler extensions over $\overline{\mathbb{K}}(z)$ behave just as in characteristic zero.

Theorem 1.7. *Let $d \geq 2$ be an integer such that $p \nmid d$. Then, a d-Mahler function $f(z) \in \mathbb{K}\{z\}$ over $\mathbb{K}(z)$ is either transcendental or in $\mathbb{K}(z)$.*

The particular case where f is the solution of a Mahler equation of order 1 can be found in [22, Chapitre 1].

Corollary 1.8. *Let $d \geq 2$ be an integer such that $p \nmid d$. Then, a d-Mahler extension over $\overline{\mathbb{K}}(z)$ is always regular over $\overline{\mathbb{K}}(z)$.*

The present paper is organized as follows. Section 2 is dedicated to the proof of Theorem 1.3. We follow the same approach as P. Philippon [21] and B. Adamczewski and C. Faverjon [2]. In Section 3 we prove Theorem 1.6. Section 4 is devoted to the proof of Theorem 1.7 and Corollary 1.8. We follow an approach of J. Roques [23] dealing with the theory of smooth projective curves in $\mathbb{P}^1(C)$ and an argument from B. Adamczewski and C. Faverjon [2]. In Section 5, we prove Corollary 1.5. Finally, in Section 6 we give an application of Theorem 1.3 and provide, in the case where $p \mid d$, examples of regular and non-regular d-Mahler extensions.

2. Proof of Theorem 1.3

Before going through the proof of Theorem 1.3, let us introduce some definitions and recall some results. Let L be a field. If $\mathfrak{q}$ is a prime ideal of $L[X_1, \ldots, X_n]$, we say that $\mathfrak{q}$ is absolutely prime over L if for every extension L_1 of L, the extended ideal $\mathfrak{q}L_1[X_1, \ldots, X_n]$ is still prime in $L_1[X_1, \ldots, X_n]$. We recall that $\mathfrak{q}$ is prime (resp. absolutely prime) in $L[X_1, \ldots, X_n]$ if and only if its homogenized ideal $\tilde{\mathfrak{q}}$ is prime (resp. absolutely prime) in $L[X_0, \ldots, X_n]$. Furthermore, when both are prime, they have the same height. Finally, given functions $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$, we recall that the extension $\overline{\mathbb{K}}(z)\,(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$ if and only if the ideal $\mathfrak{p}$ is absolutely prime in $\overline{\mathbb{K}}(z)\,[X_1, \ldots, X_n]$ [29, VII, Theorem 39].

2.1. **A local version of Theorem 1.3.** In this subsection, we establish an analogue of a result of P. Philippon [21, Proposition 4.4] in the framework of function fields. This is Corollary 2.2 below. We deduce this statement from the more general result stated in Proposition 2.1 in the vein of [2, Proposition 3.1].

Proposition 2.1. *Let $\boldsymbol{k}$ be a valued field and let $f_1(z), \ldots, f_n(z) \in \boldsymbol{k}\{z\}$ be analytic functions on a domain $\mathcal{U} \subseteq \tilde{\boldsymbol{k}}$ which contains the origin. Let us assume that the following two properties are satisfied.*

(1) *There exists a set $S \subseteq \mathcal{U} \cap \overline{\boldsymbol{k}}$ such that, for every $\alpha \in S$, we have*

$$\operatorname{trdeg}_{\overline{\boldsymbol{k}}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \operatorname{trdeg}_{\overline{\boldsymbol{k}}(z)}\{f_1(z), \ldots, f_n(z)\}. \tag{2.1}$$

(2) *The extension $\overline{\boldsymbol{k}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\boldsymbol{k}}(z)$.*

Then, there exists a finite set $S' \subseteq S$ such that for every $\alpha \in S \setminus S'$ and for every polynomial $P(X_1, \ldots, X_n) \in \overline{\boldsymbol{k}}[X_1, \ldots, X_n]$ of total degree N in $X_1, \ldots, X_n$ such that

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

there exists a polynomial $Q(z, X_1, \ldots, X_n) \in \overline{\boldsymbol{k}}[z][X_1, \ldots, X_n]$ of total degree N in $X_1, \ldots, X_n$ such that

$$Q(z, f_1(z), \ldots, f_n(z)) = 0$$

and

$$Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).$$

Corollary 2.2. *Let $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ be functions satisfying* (1.2) *and such that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$. Then, there exists ρ with $0 < \rho < 1$ such that for every $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < \rho$, and for every polynomial $P(X_1, \ldots, X_n) \in \overline{\mathbb{K}}[X_1, \ldots, X_n]$ of total degree N in $X_1, \ldots, X_n$ such that*

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

there exists a polynomial $Q(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ of total degree N in $X_1, \ldots, X_n$ such that

$$Q(z, f_1(z), \ldots, f_n(z)) = 0$$

and

$$Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).$$

Proof of Proposition 2.1. First, note that the conclusion of Proposition 2.1 is equivalent to

$$\mathrm{ev}_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]) = \tilde{\mathfrak{p}}_\alpha,$$

for all but finitely many $\alpha \in S$.

Thus proving Proposition 2.1 is the same as proving that $\mathrm{ev}_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n])$ is a prime ideal of the same height as $\tilde{\mathfrak{p}}_\alpha$ for all but finitely many $\alpha \in S$. To do so, we notice that the ring $\overline{\boldsymbol{k}}(z)[f_1(z), \ldots, f_n(z)]$ is an integral (because $\mathcal{U}$ is a domain) finitely generated $\overline{\boldsymbol{k}}(z)$-algebra. Hence applying results from commutative algebra (which only rely on these two properties and hold over any base field; see for example [9]), we get

$$\begin{aligned}
\mathrm{trdeg}_{\overline{\boldsymbol{k}}(z)}\{f_1(z), \ldots, f_n(z)\} &= \dim\left(\overline{\boldsymbol{k}}(z)[f_1(z), \ldots, f_n(z)]\right) \\
&= \dim\left(\overline{\boldsymbol{k}}(z)[X_1, \ldots, X_n]/\mathfrak{p}\right) \\
&= \dim\left(\overline{\boldsymbol{k}}(z)[X_1, \ldots, X_n]\right) - \mathrm{ht}(\mathfrak{p}) \\
&= \dim\left(\overline{\boldsymbol{k}}(z)[X_0, \ldots, X_n]\right) - \mathrm{ht}(\tilde{\mathfrak{p}}) - 1.
\end{aligned}$$

Let $\alpha \in S$. As $\overline{\boldsymbol{k}}[f_1(\alpha), \ldots, f_n(\alpha)]$ is an integral finitely generated $\overline{\boldsymbol{k}}$-algebra, we obtain in the same way that

$$\mathrm{trdeg}_{\overline{\boldsymbol{k}}}\{f_1(\alpha), \ldots, f_n(\alpha)\} = \dim\left(\overline{\boldsymbol{k}}[X_0, \ldots, X_n]\right) - \mathrm{ht}(\tilde{\mathfrak{p}}_\alpha) - 1.$$

By assumption, we get

$$\mathrm{ht}(\tilde{\mathfrak{p}}) = \mathrm{ht}(\tilde{\mathfrak{p}}_\alpha). \tag{2.2}$$

Thus, proving Proposition 2.1 is now equivalent to proving that $\mathrm{ev}_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n])$ is a prime ideal of the same height as $\tilde{\mathfrak{p}}$ for all but finitely many $\alpha \in S$. First, as, by assumption, the extension $\overline{\boldsymbol{k}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\boldsymbol{k}}(z)$, the ideal $\mathfrak{p}$ is absolutely prime over $\overline{\boldsymbol{k}}(z)[X_1, \ldots, X_n]$ [29, VII, Theorem 39]. Therefore, as recalled earlier, $\tilde{\mathfrak{p}}$ is absolutely prime over $\overline{\boldsymbol{k}}(z)[X_0, \ldots, X_n]$. Now, a result from W. Krull [15, Satz 16], which holds for any base field, leads to the existence of a finite set $S' \subseteq S$ such that for every $\alpha \in S \setminus S'$, the ideal $\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]\right)$ is absolutely prime over $\overline{\boldsymbol{k}}[X_0, \ldots, X_n]$. In particular, it is a prime ideal. Finally, let $\alpha \in S \setminus S'$. It remains to prove that

$$\mathrm{ht}\left(\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]\right)\right) = \mathrm{ht}\left(\tilde{\mathfrak{p}}\right). \tag{2.3}$$

This is implied by a result from the same article of W. Krull [15, Satz 13], involving Chow forms. Nevertheless, we can also prove (2.3) without this concept, following the approach of Y. Nesterenko and A. Shidlovskii in [18]. This way is more elementary and interesting for the proof of Theorem 1.6 (see Remark 3.1). That is why we present it here.

We first notice that

$$\mathrm{ev}_\alpha(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]) \subseteq \tilde{\mathfrak{p}}_\alpha. \tag{2.4}$$

It follows that

$$\begin{aligned}
\mathrm{ht}\left(\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]\right)\right) &\leq \mathrm{ht}\left(\tilde{\mathfrak{p}}_\alpha\right) \\
&= \mathrm{ht}\left(\tilde{\mathfrak{p}}\right) \quad \text{by (2.2).}
\end{aligned}$$

In order to prove the converse inequality, we use a result of D. Hilbert (see for example [29, VII, Theorems 41 and 42]). We give a detailed account here because we did not find a reference in print. We reproduce an argument due to C. Faverjon (unpublished). We first introduce the following definitions, according to [18].

Definition 2.3.

(1) For every $N \in \mathbb{N}$ and every homogeneous ideal I of $\overline{\boldsymbol{k}}[X_0, \dots, X_n]$, let us set

$$\mathcal{M}_I(N) = \mathrm{vect}_{\overline{\boldsymbol{k}}}\{[P]_I, P \in \overline{\boldsymbol{k}}[X_0, \dots, X_n], \text{ homogeneous of degree } N\},$$

where $[P]_I$ stands for the congruence class of P modulo I.

(2) For every $N \in \mathbb{N}$ and every homogeneous ideal J of $\overline{\boldsymbol{k}}(z)[X_0, \dots, X_n]$, let us set

$$\mathcal{L}_J(N) = \mathrm{vect}_{\overline{\boldsymbol{k}}(z)}\{[Q]_J, Q \in \overline{\boldsymbol{k}}(z)[X_0, \dots, X_n], \text{ homogeneous of degree } N\},$$

where $[Q]_J$ stands for the congruence class of Q modulo J.

Now, let us set

$$\dim_{\overline{\boldsymbol{k}}}\left(\mathcal{M}_{\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \dots, X_n]\right)}(N)\right) = \phi(N)$$

and

$$\dim_{\overline{\boldsymbol{k}}(z)}\left(\mathcal{L}_{\tilde{\mathfrak{p}}}(N)\right) = \psi(N).$$

Then, we recall the following result.

Theorem 2.4 (D. Hilbert)**.** *For every integer $N \geq 0$, the quantities $\phi(N)$ and $\psi(N)$ are finite. Moreover, for every N big enough, they are polynomials in N, and there exist $a, b > 0$ such that*

$$\begin{cases} \phi(N) & \sim_{N \to +\infty} aN^{n - ht\left(ev_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \dots, X_n]\right)\right)}, \\ \psi(N) & \sim_{N \to +\infty} bN^{n - ht(\tilde{\mathfrak{p}})}. \end{cases} \tag{2.5}$$

With this theorem in hand, we only need to prove that

$$\phi(N) \leq \psi(N) \tag{2.6}$$

for N large enough. We now set the following definition. For every $\alpha \in \overline{\boldsymbol{k}}$, we let R_α denote the localization of the ring $\overline{\boldsymbol{k}}[z]$ at the ideal $(z - \alpha)$, in other words, the subfield of $\overline{\boldsymbol{k}}(z)$ consisting of rational fractions without pole at $z = \alpha$. Then, the result [18, Lemma 3] furnishes polynomials $b_1(z), \dots, b_{\psi(N)}(z) \in \overline{\boldsymbol{k}}[z][X_0, \dots, X_n]$ such that $\mathcal{B} = \{[b_1(z)]_{\tilde{\mathfrak{p}}}, \dots, [b_{\psi(N)}(z)]_{\tilde{\mathfrak{p}}}\}$ is an α-basis of $\mathcal{L}_{\tilde{\mathfrak{p}}}(N)$ over $\overline{\boldsymbol{k}}(z)$. That is, the following two properties are satisfied.

(1) $\mathcal{B}$ is a $\overline{\boldsymbol{k}}(z)$-basis of $\mathcal{L}_{\tilde{\mathfrak{p}}}(N)$.
(2) Every residue modulo $\tilde{\mathfrak{p}}$ of a homogeneous polynomial of degree N in $R_\alpha[X_0, \dots, X_n]$ is a linear combination of $[b_1(z)]_{\tilde{\mathfrak{p}}}, \dots, [b_{\psi(N)}(z)]_{\tilde{\mathfrak{p}}}$, with coefficients in R_α.

This result is used by Y. Nesterenko and A. Shidlovskii for the field $\mathbb{C}$ instead of $\overline{\boldsymbol{k}}$. In our case, we can, as these authors, notice that the finite set of residues modulo $\tilde{\mathfrak{p}}$ of all monomials

$$X_0^{i_0} \cdots X_n^{i_n}$$

of degree $i_0 + \cdots + i_n = N$ generates $\mathcal{L}_{\tilde{\mathfrak{p}}}(N)$ over $\overline{\boldsymbol{k}}(z)$ and satisfies property (2). Among all such finite sets which generate $\mathcal{L}_{\tilde{\mathfrak{p}}}(N)$ and satisfy property (2), let us consider a set $\mathcal{S} = \{S_1(z), \dots, S_s(z)\}$ whose cardinality is minimal. If $\mathcal{S}$ does not satisfy property (1), there exist coprime polynomials $T_1(z), \ldots, T_s(z) \in \overline{\boldsymbol{k}}[z]$ such that

$$\sum_{l=1}^{s} T_l(z) S_l(z) = 0.$$

Without loss of generality, we may assume that $T_s(\alpha) \neq 0$. It follows that

$$S_s(z) = -\sum_{l=1}^{s-1} \frac{T_l(z)}{T_s(z)} S_l(z).$$

This contradicts the minimality of s. Thus, [18, Lemma 3] remains true in our framework.

Remark 2.5. We see that the proof guarantees that we can choose an α-basis of $\mathcal{L}_{\tilde{\mathfrak{p}}}(N)$ over $\overline{\boldsymbol{k}}(z)$ among the set of residues modulo $\tilde{\mathfrak{p}}$ of all monic monomials

$$X_0^{i_0} \cdots X_n^{i_n}$$

of degree $i_0 + \cdots + i_n = N$.

Now, we are going to show that the family

$$\left\{ [\mathrm{ev}_\alpha(b_i(z))]_{\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0,\ldots,X_n]\right)} \right\}_{1 \leq i \leq \psi(N)}$$

generates $\mathcal{M}_{\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0,\ldots,X_n]\right)}(N)$ over $\overline{\boldsymbol{k}}$. Let $P(X_0, \ldots, X_n) \in \overline{\boldsymbol{k}}[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree N. As $\overline{\boldsymbol{k}} \subseteq R_\alpha$, there exist elements $r_1, \ldots, r_{\psi(N)} \in R_\alpha$ such that

$$P(X_0, \ldots, X_n) - \sum_{i=1}^{\psi(N)} r_i b_i(z) \in \tilde{\mathfrak{p}}. \tag{2.7}$$

Observe that

$$P(X_0, \ldots, X_n) - \sum_{i=1}^{\psi(N)} r_i b_i(z) \in \tilde{\mathfrak{p}} \cap R_\alpha[X_0, \ldots, X_n].$$

Then, let us apply $\mathrm{ev}_\alpha(.)$ to (2.7). We get

$$\begin{aligned} P(X_0, \ldots, X_n) - \sum_{i=1}^{\psi(N)} \mathrm{ev}_\alpha(r_i)\mathrm{ev}_\alpha(b_i(z)) &\in \mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap R_\alpha[X_0, \ldots, X_n]\right) \\ &= \mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0, \ldots, X_n]\right). \end{aligned}$$

Therefore the family

$$\left\{ [\mathrm{ev}_\alpha(b_i(z))]_{\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0,\ldots,X_n]\right)} \right\}_{1 \leq i \leq \psi(N)}$$

generates $\mathcal{M}_{\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}} \cap \overline{\boldsymbol{k}}[z][X_0,\ldots,X_n]\right)}(N)$ over $\overline{\boldsymbol{k}}$. Hence, we obtain (2.6), and Proposition 2.1 is proved.

□

We now deduce Corollary 2.2.

Proof of Corollary 2.2. First, the matrices $A(z)$ and $A^{-1}(z)$ only have finitely many poles. Then, there exists ρ_0 with $0 < \rho_0 < 1$ such that for every $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < \rho_0$, α is regular with respect to system (1.2) and all the $f_i(z)$'s are analytic at α. Now, in Proposition 2.1, take S to be the set of all $\alpha \in \overline{\mathbb{K}}$ such that $0 < |\alpha| < \rho_0$. Assumption (1) of Proposition 2.1 is guaranteed by Theorem 1.1, assumption (2) is satisfied, and Corollary 2.2 follows from Proposition 2.1. □

2.2. **Proof of the inhomogeneous counterpart of Theorem 1.3.** In this section, we use the same approach as P. Philippon [21] to obtain the following inhomogeneous counterpart of Theorem 1.3 from Corollary 2.2.

Proposition 2.6. *We continue with the assumptions of Theorem* 1.1*. Let us assume further that the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$.*

Then, for every polynomial $P(X_1, \ldots, X_n) \in \overline{\mathbb{K}}[X_1, \ldots, X_n]$ of total degree N in $X_1, \ldots, X_n$ such that

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0,$$

there exists a polynomial $Q(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ of total degree N in $X_1, \ldots, X_n$ such that

$$Q(z, f_1(z), \ldots, f_n(z)) = 0$$

and

$$Q(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).$$

Proof. Let us keep the assumptions of Proposition 2.6. Let $P(X_1, \ldots, X_n) \in \overline{\mathbb{K}}[X_1, \ldots, X_n]$ be a polynomial of total degree N in $X_1, \ldots, X_n$ such that

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0. \tag{2.8}$$

Let us consider ρ from Corollary 2.2 and $r \in \mathbb{N}$ such that $0 < |\alpha^{d^r}| < \rho$. We can derive from d-Mahler system (1.2) the equality

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} = B(z) \begin{pmatrix} f_1(z^{d^r}) \\ \vdots \\ f_n(z^{d^r}) \end{pmatrix}, \tag{2.9}$$

where

$$B(z) = A^{-1}(z)A^{-1}(z^d) \cdots A^{-1}(z^{d^{r-1}}).$$

As α is regular for system (1.2), it is neither a pole of $B(z)$ nor a pole of $B^{-1}(z)$. Then, let us set $z = \alpha$ in (2.9). We obtain

$$\begin{pmatrix} f_1(\alpha) \\ \vdots \\ f_n(\alpha) \end{pmatrix} = B(\alpha) \begin{pmatrix} f_1(\alpha^{d^r}) \\ \vdots \\ f_n(\alpha^{d^r}) \end{pmatrix}.$$

Now, let us set

$$Q(X_1, \ldots, X_n) = P(\langle B_1(\alpha), \overline{X}\rangle, \ldots, \langle B_n(\alpha), \overline{X}\rangle),$$

where, for every $i \in \{1, \ldots, n\}$, we let $B_i(z)$ denote the i-th row of the matrix $B(z)$, $\overline{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$, and $\langle ., . \rangle$ refers to the classical scalar product on $\overline{\mathbb{K}}\{z\}^n$. We get

$$Q\left(f_1\left(\alpha^{d^r}\right), \ldots, f_n\left(\alpha^{d^r}\right)\right) = P(f_1(\alpha), \ldots, f_n(\alpha)) = 0.$$

As $B(\alpha)$ is invertible, $\deg_X(Q) = \deg_X(P) = N$. We now apply Corollary 2.2 to Q and α^{d^r}. There exists a polynomial $R(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ of degree N in $X_1, \ldots, X_n$ such that

$$R(z, f_1(z), \ldots, f_n(z)) = 0$$

and

$$R(\alpha^{d^r}, X_1, \ldots, X_n) = Q(X_1, \ldots, X_n).$$

It follows that

$$R(z^{d^r}, f_1(z^{d^r}), \ldots, f_n(z^{d^r})) = 0.$$

Now, let $B_i^{-1}(z), i = 1, \ldots, n$, denote the i-th row of the matrix $B^{-1}(z)$. Let $b(z) \in \mathbb{K}[z]$ be a polynomial such that for every $i \in \{1, \ldots, n\}$, $b(z)B_i^{-1}(z) \in \mathbb{K}[z]^n$ and for every $k \in \mathbb{N}$, α^{d^k} is not a zero of $b(z)$ (which is possible because for every $k \in \mathbb{N}$, α^{d^k} is not a pole of $A(z)$). Let us set

$$S(z, X_1, \ldots, X_n) = R\left(z^{d^r}, \langle B_1^{-1}(z), \overline{X}\rangle, \ldots, \langle B_n^{-1}(z), \overline{X}\rangle\right)\left(\frac{b(z)}{b(\alpha)}\right)^N.$$

By construction, we have

$$S(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n].$$

As $B^{-1}(z)$ is invertible, $\deg_X(S) = \deg_X(R) = N$. Besides, we obtain

$$S\left(z, f_1(z), \ldots, f_n(z)\right) = R\left(z^{d^r}, f_1\left(z^{d^r}\right), \ldots, f_n\left(z^{d^r}\right)\right)\left(\frac{b(z)}{b(\alpha)}\right)^N = 0.$$

Finally, as α is regular, we get

$$\begin{aligned}
S\left(\alpha, X_1, \ldots, X_n\right) &= R\left(\alpha^{d^r}, \langle B_1^{-1}(\alpha), \overline{X}\rangle, \ldots, \langle B_n^{-1}(\alpha), \overline{X}\rangle\right)\\
&= Q\left(\langle B_1^{-1}(\alpha), \overline{X}\rangle, \ldots, \langle B_n^{-1}(\alpha), \overline{X}\rangle\right)\\
&= P\left(\left\langle B_1(\alpha), \begin{pmatrix}\langle B_1^{-1}(\alpha), \overline{X}\rangle\\ \vdots \\ \langle B_n^{-1}(\alpha), \overline{X}\rangle\end{pmatrix}\right\rangle, \ldots, \left\langle B_n(\alpha), \begin{pmatrix}\langle B_1^{-1}(\alpha), \overline{X}\rangle\\ \vdots \\ \langle B_n^{-1}(\alpha), \overline{X}\rangle\end{pmatrix}\right\rangle\right)\\
&= P\left(X_1, \ldots, X_n\right).
\end{aligned}$$

Thus, we have found a polynomial $S(z, X_1, \ldots, X_n) \in \overline{\mathbb{K}}[z][X_1, \ldots, X_n]$ of degree N in $X_1, \ldots, X_n$ such that

$$S(z, f_1(z), \ldots, f_n(z)) = 0$$

and

$$S(\alpha, X_1, \ldots, X_n) = P(X_1, \ldots, X_n).$$

The inhomogeneous counterpart of Theorem 1.3 is proved. □

2.3. **End of the proof of Theorem 1.3.** The first part of the proof of Theorem 1.3 consists of showing the following analogue of [2, Théorème 4.1].

Theorem 2.7. *Under the assumptions of Theorem* 1.3*, we have*

$$Rel_{\overline{\mathbb{K}}}(f_1(\alpha), \ldots, f_n(\alpha)) = ev_\alpha(Rel_{\overline{\mathbb{K}}(z)}(f_1(z), \ldots, f_n(z))), \tag{2.10}$$

where for a field L and elements $u_1, \ldots, u_n$ of an L-vector space, we let $Rel_L(u_1, \ldots, u_n)$ denote the set of linear relations over L between the u_i's.

We do not reproduce the proof of Theorem 2.7. It can be proved as in [2] by induction on the dimension of $\mathrm{Rel}_{\overline{\mathbb{K}}(z)}(f_1(z),\dots,f_n(z))$. However, we give here more details about how to deduce Theorem 1.3 from Theorem 2.7 (also see [3]). Let $P(X_1,\dots,X_n)\in\overline{\mathbb{K}}[X_1,\dots,X_n]$ be homogeneous of degree N in $X_1,\dots,X_n$ such that

$$P(f_1(\alpha),\dots,f_n(\alpha))=0. \tag{2.11}$$

Let $\mathcal{G}_N$ denote the set of all monic monomials of degree N in $f_1(z),\dots,f_n(z)$. Then, (2.11) can be seen as a linear relation over $\overline{\mathbb{K}}$ between specializations at $z=\alpha$ of elements of $\mathcal{G}_N$. Our aim is to show that the elements of $\mathcal{G}_N$ satisfy a d-Mahler system for which α is still regular and apply Theorem 2.7 to the functions of $\mathcal{G}_N$ and P.

To do so, in what follows, we define by induction on N, n vectors $M_N^1(z),\dots,$ $M_N^n(z)$ which satisfy the following properties.

(1) For every $i\in\{1,\dots,n\}$, $M_N^i(z)$ is composed of n^{N-1} rows. We write

$$M_N^i(z)=\begin{pmatrix}L_{N,1}^i(z)\\ \vdots\\ L_{N,n^{N-1}}^i(z)\end{pmatrix}.$$

(2) For all $i\in\{1,\dots,n\}$, $j\in\{1,\dots,n^{N-1}\}$, $L_{N,j}^i(z)\in\mathcal{G}_N$.
(3) $\mathcal{G}_N\subseteq\{L_{N,j}^i(z)\}_{i,j}$.

Let us set

$$M_N(z)=\begin{pmatrix}M_N^1(z)\\ \vdots\\ M_N^n(z)\end{pmatrix}. \tag{2.12}$$

This is a vector of n^N rows of elements of $\mathcal{G}_N$. Let us define (2.12) by induction on N in the following way.

(a) For every $i\in\{1,\dots,n\}$,

$$M_1^i(z)=f_i(z).$$

(b) For all $N\geq 2$, $i\in\{1,\dots,n\}$,

$$M_N^i(z)=M_{N-1}(z)f_i(z)=\begin{pmatrix}M_{N-1}^1(z)f_i(z)\\ \vdots\\ M_{N-1}^n(z)f_i(z)\end{pmatrix}.$$

We see that this definition allows $M_N(z)$ to satisfy properties (1)-(3) for every $N\geq 1$.

Now, we have the following lemma.

Lemma 2.8. *The elements of $\mathcal{G}_N$ satisfy the d-Mahler system*

$$M_N(z^d)=A^{\otimes N}(z)M_N(z), \tag{2.13}$$

where $\otimes$ stands for the Kronecker product.

Proof. We prove Lemma 2.8 by induction on N. For $N=1$, (2.13) holds. Now, let us assume that (2.13) is satisfied at the rank $N-1$. Let us set $A(z)=(a_{i,j}(z))_{i,j}$ for the matrix of d-Mahler system (1.2). Then, we have for every $i\in\{1,\dots,n\}$,

$$\begin{aligned} M_N^i(z^d) &= M_{N-1}(z^d)f_i(z^d)\\ &= A^{\otimes N-1}(z)M_{N-1}(z)f_i(z^d) \quad \text{by assumption}\\ &= A^{\otimes N-1}(z)M_{N-1}(z)\sum_{j=1}^{n}a_{i,j}(z)f_j(z)\\ &= \sum_{j=1}^{n}\left(a_{i,j}(z)A^{\otimes N-1}(z)\right)M_N^j(z). \end{aligned} \tag{2.14}$$

If we cut the rows of the matrix $A^{\otimes N}(z)$ from top to bottom into n blocks of n^{N-1} rows, (2.14) corresponds to the product of the i-th block of $A^{\otimes N}(z)$ by $M_N(z)$. This implies Lemma 2.8. □

We are now able to end the proof of Theorem 1.3.

End of the proof of Theorem 1.3. By a property of the Kronecker product (see for example [12]), the coefficients of $A^{\otimes N}(z)$ are products of elements of $A(z)$, and we have, for every $N\geq 2$,

$$\det\left(A^{\otimes N}(z)\right)=\det\left(A(z)\right)^{nN}.$$

We deduce that α is still a regular number for d-Mahler system (2.13). On the other hand, $\overline{\mathbb{K}}(z)(f_1(z),\dots,f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$ and

$$\overline{\mathbb{K}}(z)(\mathcal{G}_N)\subseteq\overline{\mathbb{K}}(z)(f_1(z),\dots,f_n(z)).$$

Then, by [9, Corollary A1.6], $\overline{\mathbb{K}}(z)(\mathcal{G}_N)$ is separable over $\overline{\mathbb{K}}(z)$. It follows that $\overline{\mathbb{K}}(z)(\mathcal{G}_N)$ is regular over $\overline{\mathbb{K}}(z)$. Hence, Theorem 1.3 follows from Theorem 2.7. □

We end this section with the following remark, which allows us to consider Theorem 1.3 from another point of view.

Remark 2.9. Let us keep the assumptions of Theorem 1.1. Then, the regularity of

$$\overline{\mathbb{K}}(z)(f_1(z),\dots,f_n(z))$$

in the assumptions of Theorem 1.3 can be replaced by

($*$) $\mathrm{ev}_\alpha\left(\tilde{\mathfrak{p}}\cap\overline{\mathbb{K}}[z][X_0,\dots,X_n]\right)$ is prime in $\overline{\mathbb{K}}[X_0,\dots,X_n]$.

Indeed, if ($*$) is satisfied, we can reproduce the proof of Proposition 2.1 from (2.3) to the end to show that Proposition 2.1 holds. Then, Corollary 2.2 holds, and it follows from Sections 2.2 and 2.3 that Theorem 1.3 holds. Reciprocally, if Theorem 1.3 holds, we have $\mathrm{ev}_\alpha(\tilde{\mathfrak{p}}\cap\overline{\mathbb{K}}[z][X_0,\dots,X_n])=\tilde{\mathfrak{p}}_\alpha$, and ($*$) is satisfied.

3. Proof of Theorem 1.6

Let us keep the notation and assumptions of Theorem 1.6. We recall that R_α is the localization of the ring $\overline{\boldsymbol{k}}[z]$ at the ideal $(z-\alpha)$. Before going through the proof of Theorem 1.6, let us make a remark in the vein of Remark 2.5 and recall basic facts about Cartier operators, along with a result of S. Mac Lane concerning separability. Let $N \in \mathbb{N}$. We set

$$\mathcal{G} = \mathrm{vect}_{\overline{\boldsymbol{k}}(z)}\{Q(f_1(z), \ldots, f_n(z)), Q \in \overline{\boldsymbol{k}}(z)[X_1, \ldots, X_n], \text{homogeneous}, \\ \deg_X(Q) \leq N\}.$$

Remark 3.1. We can prove, in the same way as in the proof of [18, Lemma 3], that there exist monic monomials $M_l(X_1, \ldots, X_n)$, with $\deg_X(M_l) \leq N$, $l = 1, \ldots, s$, such that the family $\{M_l(f_1(z), \ldots, f_n(z))\}_l$ is a basis of $\mathcal{G}$ which satisfies the following property.

($*_\alpha$) For every $P(X_1, \ldots, X_n) \in R_\alpha[X_1, \ldots, X_n]$, there exist $P_1(z), \ldots, P_s(z) \in R_\alpha$ such that

$$P(f_1(z), \ldots, f_n(z)) = \sum_{l=1}^{s} P_l(z) M_l(f_1(z), \ldots, f_n(z)).$$

If $\boldsymbol{k}$ has characteristic p, we recall some basic facts about Cartier operators. Let $f(z) = \sum_{n=0}^{+\infty} a(n) z^n \in \tilde{\boldsymbol{k}}[[z]]$. Let $r \in \{0, \ldots, p-1\}$. The r-th Cartier operator over $\tilde{\boldsymbol{k}}[[z]]$ is defined by

$$\Lambda_r(f) = \sum_{n=0}^{+\infty} a(np+r)^{1/p} z^n.$$

Then, we recall the following result.

Proposition 3.2. *Let $f, g \in \tilde{\boldsymbol{k}}[[z]]$.*

(1) *We have*

$$f(z) = \sum_{i=0}^{p-1} \Lambda_i(f)^p z^i.$$

In particular,

$$f(z) \neq 0 \Rightarrow \exists i \in \{0, \ldots, p-1\}, \Lambda_i(f) \neq 0.$$

(2) *Let $i \in \{0, \ldots, p-1\}$. Then*

$$\Lambda_i(fg^p) = \Lambda_i(f) g.$$

Besides, if $\boldsymbol{k}$ has characteristic p, we let $\boldsymbol{k}^{1/p^\infty}$ denote the perfect closure of $\boldsymbol{k}$, that is, the union over n of the fields generated by the p^n-th roots of all the elements of $\boldsymbol{k}$. Finally, we recall a fundamental theorem from S. Mac Lane [17] (see also [9, Theorem A1.3]).

Theorem 3.3 (S. Mac Lane)**.** *Let $\boldsymbol{k} \subseteq L$ be a field extension. Then, this extension is separable if and only if every family $\{x_i\}_i$ of elements of L that is linearly independent over $\boldsymbol{k}$ remains linearly independent over $\boldsymbol{k}^{1/p^\infty}$.*

Then, we prove the following result.

Proposition 3.4. *Let $\boldsymbol{k}$ be a valued field. We assume that $f_1(z), \ldots, f_n(z) \in \overline{\boldsymbol{k}}\{z\}$ are analytic functions on a domain $\mathcal{U} \subseteq \tilde{\boldsymbol{k}}$ which contains the origin. Then, the extension $\overline{\boldsymbol{k}}(z)(f_1(z), \ldots, f_n(z))$ is separable over $\overline{\boldsymbol{k}}(z)$.*

Proof. If the characteristic of $\boldsymbol{k}$ is zero, the result is known. Now, let us assume that $\boldsymbol{k}$ has characteristic $p>0$. Let us assume by contradiction that $\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ is not separable over $\overline{\boldsymbol{k}}(z)$. Let us note that

$$\overline{\boldsymbol{k}}(z)^{1/p^\infty}=\bigcup_{k=0}^{+\infty}\overline{\boldsymbol{k}}\left(z^{1/p^k}\right).$$

Besides, let us set

$$\overline{\boldsymbol{k}}[z]^{1/p^\infty}=\bigcup_{k=0}^{+\infty}\overline{\boldsymbol{k}}\left[z^{1/p^k}\right].$$

By Theorem 3.3, there exist elements $g_1(z),\dots,g_m(z)\in\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ which are linearly independent over $\overline{\boldsymbol{k}}(z)$ but linearly dependent over $\overline{\boldsymbol{k}}(z)^{1/p^\infty}$. Let $D\in\overline{\boldsymbol{k}}[z][f_1(z),\dots,f_n(z)]\setminus\{0\}$ be such that

$$g_i(z)D\in\overline{\boldsymbol{k}}[z][f_1(z),\dots,f_n(z)]\quad\forall i\in\{1,\dots,m\}.$$

Then, $g_1(z)D,\dots,g_m(z)D$ are linearly independent over $\overline{\boldsymbol{k}}(z)$ but linearly dependent over $\overline{\boldsymbol{k}}(z)^{1/p^\infty}$. Hence, even if it means replacing each $g_i(z)$ by $g_i(z)D$, we assume that for every $i\in\{1,\dots,m\}$, $g_i(z)\in\overline{\boldsymbol{k}}[z][f_1(z),\dots,f_n(z)]$.

Then, there exist elements $G_1(z),\dots,G_m(z)\in\overline{\boldsymbol{k}}[z]^{1/p^\infty}$ not all zero such that

$$\sum_{i=1}^m G_i(z)g_i(z)=0.$$

Without loss of generality, we may assume that $G_m(z)\neq 0$. On the other hand, there exists an integer $\mu\geq 1$ such that $G_i(z)^{p^\mu}\in\overline{\boldsymbol{k}}[z]$ for every $i\in\{1,\dots,m\}$. Then, we have

$$\sum_{i=1}^m G_i(z)^{p^\mu}g_i(z)^{p^\mu}=0. \tag{3.1}$$

Let us note that for every $i\in\{1,\dots,m\}$,

$$G_i(z)^{p^\mu},g_i(z)\in\overline{\boldsymbol{k}}[z][f_1(z),\dots,f_n(z)]\subseteq\overline{\boldsymbol{k}}\{z\}.$$

Now, let us choose for every integer $j\in\{1,\dots,\mu\}$ a Cartier operator $\Lambda^{(j)}$ such that

$$\Lambda^{(\mu)}\circ\cdots\circ\Lambda^{(1)}(G_m(z)^{p^\mu})\neq 0.$$

We apply $\Lambda:=\Lambda^{(\mu)}\circ\cdots\circ\Lambda^{(1)}$ to (3.1) and get

$$\sum_{i=1}^m \Lambda(G_i(z)^{p^\mu})g_i(z)=0. \tag{3.2}$$

Then, (3.2) is a non-trivial linear relation between the $g_i(z)$'s over $\overline{\boldsymbol{k}}(z)$, a contradiction. Proposition 3.4 is thus proved. □

Before proving Theorem 1.6, we introduce some definitions. We recall that we let $\boldsymbol{k}$ denote a valued field and $\boldsymbol{k}_c$ its completion. Its valuation extends uniquely to $\overline{\boldsymbol{k}_c}$, and we let $\tilde{\boldsymbol{k}}$ denote the completion of $\overline{\boldsymbol{k}_c}$ with respect to this valuation. Now, let $\mathcal{U}\subseteq\tilde{\boldsymbol{k}}$ be a domain. We say that a function is meromorphic on $\mathcal{U}$ if there exists a (possibly empty) discrete closed subset $\mathcal{P}$ of $\mathcal{U}$ such that $f(z)$ is analytic on $\mathcal{U}\setminus\mathcal{P}$, and each element of $\mathcal{P}$ is a pole of $f(z)$. Then, for every $\alpha\in\mathcal{P}$, $f(z)$ admits a convergent Laurent power series expansion in a punctured neighbourhood of α with coefficients in $\tilde{\boldsymbol{k}}$ of the form $\sum_{n=-N}^{+\infty}a_n(z-\alpha)^n$, for some $N\in\mathbb{N}$. We notice that if $\{f_i(z)\}_{1\leq i\leq n}\subset \mathbb{K}\{z\}$ satisfies system (1.2), if $0<|\alpha|<1$, and if for every $k\in\mathbb{N}$ the number α^{d^k} is not a pole of $A^{-1}(z)$, then the $f_i(z)$ are well-defined at α and $\{f_i(z)\}_{1\leq i\leq n}\subset C\{z-\alpha\}$.

We are now able to prove Theorem 1.6.

Proof of Theorem 1.6. Let us assume that the extension $\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ is not regular over $\overline{\boldsymbol{k}}(z)$. We recall that $\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ is regular over $\overline{\boldsymbol{k}}(z)$ if

(1) $\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ is separable over $\overline{\boldsymbol{k}}(z)$,
(2) every element of $\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ that is algebraic over $\overline{\boldsymbol{k}}(z)$ belongs to $\overline{\boldsymbol{k}}(z)$.

By Proposition 3.4, we only have to prove that the conclusion of Theorem 1.6 holds when there exists an element of $\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))$ that is algebraic over $\overline{\boldsymbol{k}}(z)$ but does not belong to $\overline{\boldsymbol{k}}(z)$.

Thus, let us assume that there exists an element $a(z)\in\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))\cap\overline{\overline{\boldsymbol{k}}(z)}\setminus\overline{\boldsymbol{k}}(z)$. We can write

$$a(z)=\frac{P(f_1(z),\dots,f_n(z))}{Q(f_1(z),\dots,f_n(z))}, \tag{3.3}$$

where $P(X_1,\dots,X_n),Q(X_1,\dots,X_n)\in\overline{\boldsymbol{k}}[z][X_1,\dots,X_n]$ are polynomials of total degree less than or equal to some integer $N\geq 0$.

We recall that we let $\mathcal{G}$ denote the $\overline{\boldsymbol{k}}(z)$-vector space generated by all homogeneous polynomials of degree less than or equal to N in $f_1(z),\dots,f_n(z)$.

By Remark 3.1, there exist monic monomials $M_l(X_1,\dots,X_n)$, with $\deg_X(M_l)\leq N$, $l=1,\dots,s$, such that the family $\{M_l(\{f_i(z)\})\}_l$ is a basis of $\mathcal{G}$ over $\overline{\boldsymbol{k}}(z)$ which satisfies property $(*_\alpha)$ of Remark 3.1. Then, (3.3) turns into

$$a(z)=\frac{N_1(z)M_1(\{f_i(z)\})+\cdots+N_s(z)M_s(\{f_i(z)\})}{D_1(z)M_1(\{f_i(z)\})+\cdots+D_s(z)M_s(\{f_i(z)\})}, \tag{3.4}$$

where for every $l\in\{1,\dots,s\}$, $N_l(z),D_l(z)\in\overline{\boldsymbol{k}}[z]$.

We can rewrite (3.4) as

$$F_1(z)M_1(\{f_i(z)\})+\cdots+F_s(z)M_s(\{f_i(z)\})=0, \tag{3.5}$$

where for every $l\in\{1,\dots,s\}$, $F_l(z)=D_l(z)a(z)-N_l(z)$.

We may assume without loss of generality that for every l, $F_l(z)\in\tilde{\boldsymbol{k}}\{z-\alpha\}$. Indeed, on the one hand, as the functions $f_1(z),\dots,f_n(z)\in\tilde{\boldsymbol{k}}\{z-\alpha\}$, $a(z)$ can be expressed as a Laurent power series at the point $z=\alpha$. If $a(z)\notin\tilde{\boldsymbol{k}}\{z-\alpha\}$, writing $u>0$ the order of the pole of $a(z)$ at $z=\alpha$, we could replace $a(z)$ by the function

$$(z-\alpha)^u a(z)\in\overline{\boldsymbol{k}}(z)(f_1(z),\dots,f_n(z))\cap\overline{\overline{\boldsymbol{k}}(z)}\setminus\overline{\boldsymbol{k}}(z),$$

which has no pole at $z=\alpha$. Therefore, we can assume that $a(z)\in\tilde{\boldsymbol{k}}\{z-\alpha\}$. Then, as for every $l\in\{1,\dots,s\}$, $N_l(z),D_l(z)\in\overline{\boldsymbol{k}}[z]$, we get that $F_l(z)\in\tilde{\boldsymbol{k}}\{z-\alpha\}$.

Now, let us notice that we can assume without loss of generality that

$$\exists l_0\in\{1,\dots,s\}, F_{l_0}(\alpha)\neq 0. \tag{3.6}$$

Indeed, $F_l(z) \in \tilde{\boldsymbol{k}}\{z-\alpha\}$. Therefore, if (3.6) is not satisfied, let $v > 0$ denote the minimal order at α as a zero of the functions $F_l(z)$. Then, instead of (3.5), we could consider the equation

$$G_1(z)M_1(\{f_i(z)\}) + \cdots + G_s(z)M_s(\{f_i(z)\}) = 0, \tag{3.7}$$

where for every $l \in \{1, \ldots, s\}$, $G_l(z) = \frac{F_l(z)}{(z-\alpha)^v} \in \tilde{\boldsymbol{k}}\{z-\alpha\}$. The functions G_l satisfy (3.6). Hence, even if it means replacing (3.5) by (3.7), we assume that (3.6) holds.

Then, we have

$$F_1(\alpha)M_1(\{f_i(\alpha)\}) + \cdots + F_s(\alpha)M_s(\{f_i(\alpha)\}) = 0. \tag{3.8}$$

Hence, setting

$$P\left(X_1, \ldots, X_n\right) = \sum_{l=1}^{s} F_l(\alpha)M_l\left(X_1, \ldots, X_n\right),$$

we get

$$P(f_1(\alpha), \ldots, f_n(\alpha)) = 0. \tag{3.9}$$

Let us assume by contradiction that the relation (3.9) lifts into a functional relation over $\overline{\boldsymbol{k}}(z)$. Let $N' \leq N$ denote the total degree of $P\left(X_1, \ldots, X_n\right)$. Then, there exists a polynomial

$$Q\left(z, X_1, \ldots, X_n\right) \in \overline{\boldsymbol{k}}[z][X_1, \ldots, X_n],$$

of total degree N' in $X_1, \ldots, X_n$, such that

$$Q\left(z, f_1(z), \ldots, f_n(z)\right) = 0 \tag{3.10}$$

and

$$Q\left(\alpha, X_1, \ldots, X_n\right) = P\left(X_1, \ldots, X_n\right). \tag{3.11}$$

Let us notice that the family $\{M_l(X_1, \ldots, X_n)\}_l$ is free over $\overline{\boldsymbol{k}}(z)$. Let

$$\{N_j(X_1, \ldots, X_n)\}_{1 \leq j \leq t}$$

be a family of monic monomials such that the family

$$\{M_l(X_1, \ldots, X_n), N_j(X_1, \ldots, X_n)\}_{l,j}$$

is a basis of the $\overline{\boldsymbol{k}}(z)$-vector space spanned by all homogeneous polynomials of degree less than or equal to N in $X_1, \ldots, X_n$.

Then, we can write the polynomial $Q\left(z, X_1, \ldots, X_n\right)$ as

$$Q\left(z, X_1, \ldots, X_n\right) = \sum_{l=1}^{s} Q_l(z)M_l\left(X_1, \ldots, X_n\right) + \sum_{j=1}^{t} R_j(z)N_j\left(X_1, \ldots, X_n\right),$$

where for every l and every j, $Q_l(z), R_j(z) \in \overline{\boldsymbol{k}}[z]$. By (3.11), we have

$$\begin{cases} Q_l(\alpha) = F_l(\alpha) \quad \forall l \in \{1, \ldots, s\}, \\ R_j(\alpha) = 0 \quad \forall j \in \{1, \ldots, t\}. \end{cases} \tag{3.12}$$

Now, we have

$$\begin{aligned} 0 &= Q\left(z, f_1(z), \ldots, f_n(z)\right) \\ &= \sum_{l=1}^{s} Q_l(z)M_l\left(\{f_i(z)\}\right) + \sum_{j=1}^{t} R_j(z)N_j\left(\{f_i(z)\}\right). \end{aligned} \tag{3.13}$$

Let us remark that

$$N_j\left(X_1,\dots,X_n\right)\in\overline{\boldsymbol{k}}[X_1,\dots,X_n]\subseteq R_\alpha[X_1,\dots,X_n].$$

Hence, by Remark 3.1, we get that for every $j\in\{1,\dots,t\}$ there exist polynomials $S_{j,1}(z),\dots,S_{j,s}(z)\in R_\alpha$ such that

$$N_j\left(\{f_i(z)\}\right)=\sum_{l=1}^{s}S_{j,l}(z)M_l\left(\{f_i(z)\}\right).$$

Therefore, (3.13) turns into

$$0=\sum_{l=1}^{s}Q_l(z)M_l\left(\{f_i(z)\}\right)+\sum_{j=1}^{t}R_j(z)\sum_{l=1}^{s}S_{j,l}(z)M_l\left(\{f_i(z)\}\right).$$

Now, $\{M_l\left(f_1(z),\dots,f_n(z)\right)\}_l$ is a $\overline{\boldsymbol{k}}(z)$-basis of $\mathcal{G}$. Thus, we obtain

$$Q_l(z)+\sum_{j=1}^{t}R_j(z)S_{j,l}(z)=0\quad\forall l\in\{1,\dots,s\}$$

and

$$Q_l(\alpha)+\sum_{j=1}^{t}R_j(\alpha)S_{j,l}(\alpha)=0\quad\forall l\in\{1,\dots,s\}.$$

By (3.12), we get

$$F_l(\alpha)=0\quad\forall l\in\{1,\dots,s\},$$

which contradicts (3.6). Theorem 1.6 is proved. □

4. Proofs of Theorem 1.7 and Corollary 1.8

We prove Theorem 1.7 following the strategy of J. Roques [23]. We extend Proposition 4 and Corollary 5 of [23] for the base field C instead of $\overline{\mathbb{F}_p}$. The analogue of Proposition 4 is the following.

Proposition 4.1. *Let L be a finite extension of $C(z)$. Let $d\geq 2$ be an integer such that $p\nmid d$. Let us assume that the endomorphism ϕ_d of $C(z)$ defined by $\phi_d(P(z))=P(z^d)$ extends to a field endomorphism of L. Then, there exist a positive integer N and $z_N\in L$ such that*

(i) *$z_N^N=z$,*
(ii) *L is a purely inseparable extension of $C(z_N)$.*

Proof. We still let ϕ_d denote its extension to L. Let E denote the separable closure of $C(z)$ in L. Then, we see that for every $x\in E$, $\phi_d(x)\in E$. Hence, ϕ_d induces a field endomorphism of E. Let X denote a smooth projective curve whose function field is E (see for example [13, I.6]). Let $j : \mathbb{P}^1(C) \to \mathbb{P}^1(C)$ be the morphism of curves associated with $\phi_d : C(z) \to C(z)$, $f : X \to X$ the morphism of curves associated with the extension of ϕ_d to E, and $\varphi : X \to \mathbb{P}^1(C)$ the morphism of curves associated with the inclusion $i : C(z) \hookrightarrow E$. Then, we have the following commutative diagram:

$$(D_1) \qquad \begin{tikzcd} (x_1,x_2) \arrow[d, mapsto, "\varphi"] & X \arrow[r, "f"] \arrow[d] & X \arrow[d] & (x_1,x_2) \arrow[d, mapsto, "\varphi"'] \\ \left(1,\left(\frac{x_2}{x_1}\right)\right) & \mathbb{P}^1(C) \arrow[r] & \mathbb{P}^1(C) & \left(1,\left(\frac{x_2}{x_1}\right)\right) \\ & (x_1,x_2) \arrow[r, mapsto, "j"'] & \left(1,\left(\frac{x_2}{x_1}\right)^d\right) & \end{tikzcd}$$

Now, we prove that f satisfies the following properties.

(1) f is a separable morphism; that is, $E/\phi_d(E)$ is a separable extension.
(2) f has degree d.
(3) f is totally ramified above any point of $\varphi^{-1}(0) \cup \varphi^{-1}(\infty)$.

To prove the first assertion, it suffices to show that $E/\phi_d(C(z))$ is separable. But $\phi_d(C(z)) = C(z^d)$. As $p \nmid d$, $C(z)/C(z^d)$ is separable. Besides, by definition, $E/C(z)$ is separable. Assertion 1 follows. The second assertion can be read on the diagram (D_1). We get $\deg(f)\deg(\varphi) = \deg(\varphi)\deg(j)$. But $\deg(j) = [C(z) : \phi_d(C(z))] = d$. Finally, let us prove the last assertion. By diagram (D_1), we get $f^{-1}(\varphi^{-1}(0)) = \varphi^{-1}(0)$. Besides, as X is a smooth projective curve, by [13, II, 6.7, 6.8, Exercise 3.5], the set $\varphi^{-1}(0)$ is finite. Let $x \in \varphi^{-1}(0)$. We deduce that the set $f^{-1}(x)$ has exactly one element. Now, it follows from [27, II, Proposition 2.6] that f is totally ramified above x. The same arguments hold for $\varphi^{-1}(\infty)$, and assertion (3) is proved. Now, let g be the genus of X. We prove that $g \in \{0, 1\}$. First, let us recall the Hurwitz formula (see for example [13, IV.2.4]). If $\vartheta : W \to W$ is a finite separable morphism of curves, we have

$$-2(g(W) - 1)(n(\vartheta) - 1) \geq \sum_{P \in W} (e_P - 1), \tag{4.1}$$

where the integer $g(W) \geq 0$ is the genus of the curve W, the integer $n(\vartheta) \geq 1$ is the degree of ϑ, and the integer $e_P \geq 1$ is the ramification index of ϑ at P. Now, if $g \notin \{0, 1\}$, it follows from Hurwitz formula (4.1) that all the compositions $f^i(z), i \geq 0$, are automorphisms of the smooth projective curve X. But H. L. Schmid proved [24] that there only exist finitely many automorphisms of X when $g \geq 2$ (see also [28]). As ϕ_d has infinite order, it is the same for $f(z)$, and we get a contradiction. Hence, $g \in \{0, 1\}$. But if $g = 1$, it follows from Hurwitz formula (4.1) that f is unramified everywhere. This contradicts assertion (3). Hence, $g = 0$. Now, our goal is to prove the following lemma.

Lemma 4.2. *There exists a transcendental element u over C such that $E = C(u)$. Moreover, there exists $P(u) \in C(u)$ such that the following diagram commutes:*

$$(D_2) \qquad \begin{tikzcd}
& u^d & u \arrow[l, mapsto, "h_2"'] & \\
P(u) & C(u) & C(u) \arrow[l] & P(u) \\
z \arrow[u, mapsto, "h_1"'] & C(z) \arrow[u] & C(z) \arrow[l] \arrow[u] & z \arrow[u, mapsto, "h_1"] \\
& z^d & z \arrow[l, mapsto, "\phi_d"] &
\end{tikzcd}$$

Proof. As $g = 0$, X and $\mathbb{P}^1(C)$ are birationally equivalent (see [13, IV, 1.3.5]). By [13, I.4.5], E and $C(z)$ are isomorphic as C-algebras. Hence

$$E = C(u),$$

where $u \in E$ is transcendental over C. Now, we are going to express the morphisms h_1 and h_2 of diagram (D_2) with respect to the function field morphisms associated with the morphisms of curves of diagram (D_1). To start, applying Hurwitz formula (4.1) to f, we get that the sets $\varphi^{-1}(0)$ and $\varphi^{-1}(\infty)$ have respectively exactly one element, say a and b, and that the following property is satisfied:

(4) f is unramified outside $\{a, b\}$.

On the other hand, we notice that the following properties characterize the morphism of curves $\tilde{h_2} : X \longrightarrow X$ associated with the function field morphism h_2 of diagram (D_2) (we identify $0, \infty$ of $\mathbb{P}^1(C)$ with the corresponding elements of X via birational equivalence).

(1') $\tilde{h_2}(0) = 0$, $\tilde{h_2}(\infty) = \infty$.
(2') $\tilde{h_2}$ has degree d.
(3') $\tilde{h_2}$ is totally ramified at 0 and ∞.
(4') $\tilde{h_2}$ is unramified outside $\{0, \infty\}$.

These assertions are exactly the assertions (1)-(4) satisfied by f, except that $\{a, b\}$ is replaced by $\{0, \infty\}$. To correct it, we consider an automorphism c of X such that

$$\begin{aligned} c : X &\to X \\ a &\mapsto 0 \\ b &\mapsto \infty. \end{aligned}$$

From now on, if h is a morphism of curves, we let h^* denote the associated morphism of function fields. We deduce from properties (1)-(4) of f that the morphism cfc^{-1} satisfies properties (1')-(4'). Hence $h_2 = (cfc^{-1})^*$. Now, let $h_1 = \left(\varphi c^{-1}\right)^*$ and $P(u) \in C(u)$ be such that $\left(\varphi c^{-1}\right)^* (z) = P(u)$.

By diagram (D_1), we get the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "cfc^{-1}"] \arrow[d, "\varphi c^{-1}"'] & X \arrow[d, "\varphi c^{-1}"] \\
\mathbb{P}^1(C) \arrow[r] & \mathbb{P}^1(C) \\[-20pt]
(x_1, x_2) \arrow[r, mapsto, "j"'] & \left(1, \left(\frac{x_2}{x_1}\right)^d\right)
\end{tikzcd}$$

Lemma 4.2 follows by considering the associated morphisms of function fields. □

We are now able to conclude the proof of Proposition 4.1. Let us read diagram (D_2). On the one hand, we obtain

$$h_2 \circ h_1(z) = P(u^d),$$

and on the other hand,

$$h_1 \circ \phi_d(z) = P(u)^d.$$

Then $P(u^d) = P(u)^d$. But $p \nmid d$. Hence

$$P(u) = \lambda u^N,$$

where $N \in \mathbb{Z}$ and $\lambda^d = \lambda \in C$.

Now, let $c_1 = \left(c^{-1}\right)^*$ denote the function field automorphism associated with c^{-1}. We have

$$h_1 = \left(\varphi c^{-1}\right)^* = c_1 \varphi^* = c_1 i.$$

Let us set

$$i(z) = z = Q(u),$$

where $Q(u) \in C(u)$. Then, we get

$$\begin{aligned} \lambda u^N &= h_1(z) \\ &= c_1 i(z) \\ &= c_1(Q(u)). \end{aligned}$$

Hence

$$\begin{aligned} z &= Q(u) \\ &= c_1^{-1}\left(\lambda u^N\right) \\ &= \lambda\left(c_1^{-1}(u)\right)^N. \end{aligned}$$

Now, let μ be an N-th root of λ in C. Let us set

$$\begin{cases} z_N = \mu c_1^{-1}(u) \text{ if } N \geq 0, \\ \quad\;\; = 1/(\mu c_1^{-1}(u)) \text{ otherwise.} \end{cases}$$

In both cases, $z_N \in E$, and we obtain

$$z = z_N^{|N|}$$

and

$$E = C(u) = C(z_N).$$

Finally, as E is the separable closure of $C(z)$ in L, L is a purely inseparable extension of $C(z_N)$. Proposition 4.1 is proved. □

We deduce the analogue of Corollary 5 of [23].

Corollary 4.3. *Let $L \subset C((z))$ be a finite extension of $C(z)$. Let $d \geq 2$ be an integer such that $p \nmid d$. We assume that the endomorphism ϕ_d of $C(z)$ defined by $\phi_d(P(z)) = P(z^d)$ extends to a field endomorphism of L. Then, L is a purely inseparable extension of $C(z)$.*

Proof. We have $z_N^N = z$, with $z_N \in L \subset C((z))$. Hence, $N = 1$, $z_N = z$, and Corollary 4.3 is proved. □

We are now able to prove Theorem 1.7. To do so, we use here Cartier operators (see Proposition 3.2).

Proof of Theorem 1.7. Let us assume that $f(z)$ is algebraic over $\overline{\mathbb{K}}(z)$. Let (1.4) be the minimal inhomogeneous equation of $f(z)$ and set

$$L = C(z)\left(f(z), \ldots, f\left(z^{d^{m-1}}\right)\right).$$

We note that for every $l \geq 0$, $f\left(z^{d^l}\right)$ is algebraic over $\overline{\mathbb{K}}(z)$. It follows that L is a finite extension of $C(z)$. Besides, d-Mahler equation (1.4) guarantees that ϕ_d induces a field endomorphism of L. Then, by Corollary 4.3, L is a purely inseparable extension of $C(z)$. We deduce that there exists an integer s such that $f(z)^{p^s} \in C(z)$. Hence, there exist non-zero polynomials $A(z), B(z) \in C[z]$ such that

$$B(z)f(z)^{p^s} = A(z). \tag{4.2}$$

Now, let us choose for every integer $j \in \{1, \ldots, s\}$ a Cartier operator $\Lambda^{(j)}$ such that

$$\Lambda^{(s)} \circ \cdots \circ \Lambda^{(1)}(B(z)) \neq 0.$$

We apply $\Lambda := \Lambda^{(s)} \circ \cdots \circ \Lambda^{(1)}$ to (4.2) and get

$$\Lambda(B(z))f(z) = \Lambda(A(z)).$$

Then, $f(z) \in C(z) \cap \mathbb{K}\{z\} = \mathbb{K}(z)$, and Theorem 1.7 is proved. □

Proof of Corollary 1.8. Let $L = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ be a d-Mahler extension over $\overline{\mathbb{K}}(z)$. Without loss of generality, we can assume that $(f_1(z), \ldots, f_n(z))$ is a solution vector of system (1.2). Indeed, if not, we can insert the $f_i(z)$ into a solution vector

$$\underline{g}(z) = (f_1(z), \ldots, f_n(z), g_{n+1}(z), \ldots, g_N(z))$$

of a d-Mahler system of size $N \geq n$. Then, we have

$$L \subseteq \overline{\mathbb{K}}(z)(\underline{g}(z)).$$

Thus, if we prove that $\overline{\mathbb{K}}(z)(\underline{g}(z))$ is regular over $\overline{\mathbb{K}}(z)$, it follows from [9, Corollary A1.6] that L is regular over $\overline{\mathbb{K}}(z)$. Now, by Proposition 3.4, L is separable over $\overline{\mathbb{K}}(z)$. It thus remains to prove that every element of L that is algebraic over $\overline{\mathbb{K}}(z)$ belongs to $\overline{\mathbb{K}}(z)$. To do so, we follow the same approach as in [2]. Let E be the algebraic closure of $\overline{\mathbb{K}}(z)$ in L and $f(z) \in E$. Our aim is to prove that $f(z)$ is

d-Mahler and apply Theorem 1.7. First, it follows from system (1.2) that for every $l \geq 0$, $f(z^{d^l}) \in L$. Then, the fact that $f(z) \in E$ implies that for every $l \geq 0$, $f(z^{d^l}) \in E$. Now, it suffices to prove that E is a finite extension of $\overline{\mathbb{K}}(z)$. As L is a finitely generated extension of $\overline{\mathbb{K}}(z)$, the subextension E has the same property (see for example [16, VIII, Exercise 4]). But E is also an algebraic extension of $\overline{\mathbb{K}}(z)$. Hence, E is a finite extension of $\overline{\mathbb{K}}(z)$. It follows that $f(z)$ is d-Mahler. Thus, by Theorem 1.7, $f(z) \in \mathbb{K}(z)$, and Corollary 1.8 is proved. □

5. Proof of Corollary 1.5

We prove here Corollary 1.5.

Proof of Corollary 1.5. Let us keep the assumptions of Corollary 1.5. Let (1.5) be the minimal inhomogeneous system satisfied by $f(z)$. Let us prove the first assertion. Let us assume that $f(\alpha) \in \overline{\mathbb{K}}$. For every $i \in \{1, \ldots, m\}$, let us set

$$f_i(z) = f(z^{d^{i-1}}).$$

There exists an integer $l \geq 0$ such that α^{d^l} is regular for d-Mahler system (1.5). Therefore, by Theorem 1.3 and minimality of (1.5), the numbers $1, f_1(\alpha^{d^l}), \ldots, f_m(\alpha^{d^l})$ are linearly independent over $\overline{\mathbb{K}}$. Moreover, we can write

$$\begin{pmatrix} 1 \\ f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = A_l(z) \begin{pmatrix} 1 \\ f_1(z^{d^l}) \\ \vdots \\ f_m(z^{d^l}) \end{pmatrix}, \tag{5.1}$$

where

$$A_l(z) = A^{-1}(z)A^{-1}(z^d) \cdots A^{-1}\left(z^{d^{l-1}}\right).$$

Then, we can see that α is not a pole of the matrix $A_l(z)$ of system (5.1). Indeed, otherwise, let r denote the maximum of the order of α as a zero of the denominators of the coefficients of $A_l(z)$. We get

$$(z-\alpha)^r \begin{pmatrix} 1 \\ f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = B_l(z) \begin{pmatrix} 1 \\ f_1(z^{d^l}) \\ \vdots \\ f_m(z^{d^l}) \end{pmatrix}, \tag{5.2}$$

where $B_l(z) = (z-\alpha)^r A_l(z)$ has no pole at α and is such that $B_l(\alpha) \neq 0$. Then, setting $z = \alpha$ in (5.2), we find a linear non-trivial relation between the numbers $1, f_1(\alpha^{d^l}), \ldots, f_m(\alpha^{d^l})$ which contradicts the fact that they are linearly independent over $\overline{\mathbb{K}}$. Now, if we set

$$\Lambda = \begin{pmatrix} f(\alpha) & -1 & 0 & \cdots & 0 \end{pmatrix},$$

we obtain

$$0 = \Lambda \begin{pmatrix} 1 \\ f_1(\alpha) \\ \vdots \\ f_m(\alpha) \end{pmatrix} = \Lambda A_l(\alpha) \begin{pmatrix} 1 \\ f_1(\alpha^{d^l}) \\ \vdots \\ f_m(\alpha^{d^l}) \end{pmatrix}.$$

Then, the fact that $1, f_1(\alpha^{d^l}), \ldots, f_m(\alpha^{d^l})$ are linearly independent over $\overline{\mathbb{K}}$ implies that

$$\Lambda A_l(\alpha) = 0. \tag{5.3}$$

Now, as the first coordinate of the solution vector of system (5.1) is 1, there exists a column vector $\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{K}(\alpha)^{m+1}$ of $A_l(\alpha)$ such that $u_0 \neq 0$. Then, by (5.3) we get

$$f(\alpha) = \frac{u_1}{u_0} \in \mathbb{K}(\alpha).$$

Let us prove the second statement. If α is a regular number for system (1.5), let us assume by contradiction that $f(\alpha)$ is algebraic over $\overline{\mathbb{K}}$, that is, $f(\alpha) \in \overline{\mathbb{K}}$. Then, the numbers $1, f(\alpha)$ are linearly dependent over $\overline{\mathbb{K}}$, and hence the numbers $1, f(\alpha), \ldots, f\left(\alpha^{d^{m-1}}\right)$ are linearly dependent over $\overline{\mathbb{K}}$. Then, Theorem 1.3 implies that there exists a non-trivial linear relation between the functions $1, f(z), \ldots, f\left(z^{d^{m-1}}\right)$ over $\overline{\mathbb{K}}(z)$. This contradicts the minimality of equation (1.4) and proves that $f(\alpha)$ is transcendental over $\overline{\mathbb{K}}$. $\square$

Remark 5.1. In order to prove the first statement of Corollary 1.5, we showed the existence of an integer $l \geq 0$ and a matrix $B(z) \in \mathrm{GL}_{m+1}(\mathbb{K}(z))$ such that

$$\begin{pmatrix} 1 \\ f_1(z) \\ \vdots \\ f_m(z) \end{pmatrix} = B(z) \begin{pmatrix} 1 \\ f_1(z^{d^l}) \\ \vdots \\ f_m(z^{d^l}) \end{pmatrix} \tag{5.4}$$

and

(1) the number α^{d^l} is regular for system (5.4),
(2) the number α is not a pole of $B(z)$.

This is the analogue of [6, Theorem 1.5] for E-functions and of [2, Théorème 1.10] in the particular case of linearly independent Mahler functions.

6. Examples

In this section, we illustrate Theorem 1.3 and provide examples, in the case where $p \mid d$, of regular and non-regular d-Mahler extensions.

6.1. **An application of Theorem 1.3.** Let d be an integer such that $p \nmid d$. Let us consider the system

$$\begin{pmatrix} f_1(z^d) \\ f_2(z^d) \\ f_3(z^d) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ z^{d^2} - z & 1 & -1 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{pmatrix}. \tag{6.1}$$

Let us set

$$a(z) = z + \sum_{n=1}^{+\infty} F_n z^{d^n}, \tag{6.2}$$

where F_n is the residue modulo p of the n-th Fibonacci number (with $F_1 = 1, F_2 = 1$). Then, a solution to system (6.1) is given by

$$f_1(z) = 1, f_2(z) = a(z), f_3(z) = a(z^d). \tag{6.3}$$

By Corollary 1.8, the d-Mahler extension $\mathcal{E} = \overline{\mathbb{K}}(z)(f_1(z), f_2(z), f_3(z))$ is regular over $\overline{\mathbb{K}}(z)$. The advantage of Theorem 1.3 is that we do not have to study algebraic relations between $f_1(z)$, $f_2(z)$, $f_3(z)$ to get the following result.

Proposition 6.1. *Let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$. Then, $1, a(\alpha), a(\alpha^d)$ are linearly independent over $\overline{\mathbb{K}}$.*

By Corollary 1.4, all we have to prove is the following result.

Lemma 6.2. *The functions $f_1(z), f_2(z), f_3(z)$ are linearly independent over $\overline{\mathbb{K}}(z)$.*

Proof of Lemma 6.2 does not involve difficult arguments and illustrates the interest of Theorem 1.3.

Proof. Let us assume by contradiction that there exist coprime polynomials

$$P_{-1}(z), P_0(z), P_1(z) \in \overline{\mathbb{K}}[z]$$

such that

$$P_{-1}(z) + P_0(z)a(z) + P_1(z)a(z^d) = 0.$$

It follows that

$$P_{-1}(z) + P_0(z)z + P_0(z)\sum_{n=1}^{+\infty} F_n z^{d^n} + P_1(z)z^d + P_1(z)\sum_{n=2}^{+\infty} F_{n-1} z^{d^n} = 0. \tag{6.4}$$

For every $n \geq 1$, let us set $a_n = d^n - d^{n-1}$. Then, the sequence $(a_n)_n$ is strictly increasing. Now, let us take a non-zero integer $N \in \mathbb{N}$ such that $a_n > \max(\deg(P_i))$ for every $n \geq N$. If we compare the coefficients of z^{d^N} and $z^{d^{N+1}}$, respectively, between the left- and right-hand side of (6.4), we get

$$\begin{cases} p_{0,0}F_N + p_{1,0}F_{N-1} = 0, \\ p_{0,0}F_{N+1} + p_{1,0}F_N = 0, \end{cases} \tag{6.5}$$

where $p_{0,0}, p_{1,0}$ are respectively the constant term of $P_0(z)$ and $P_1(z)$. By property of the Fibonacci sequence, the determinant $F_N^2 - F_{N-1}F_{N+1}$ of system (6.5) is equal to $(-1)^{N+1} \neq 0$. Hence, $p_{0,0} = p_{1,0} = 0$. But, by (6.4), the constant term of $P_{-1}(z)$ is equal to zero. This contradicts the fact that the P_i's are coprime. Lemma 6.2 is proved. $\square$

6.2. **Regular extensions.** If $f_1(z), \ldots, f_n(z) \in \mathbb{K}\{z\}$ are algebraically independent functions over $\overline{\mathbb{K}}(z)$, then the extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$ is regular over $\overline{\mathbb{K}}(z)$. Indeed, let us set $\mathcal{E} = \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z))$. As $(f_1(z), \ldots, f_n(z))$ is a transcendence basis of $\mathcal{E}$ over $\overline{\mathbb{K}}(z)$, $\mathcal{E}$ is separable over $\overline{\mathbb{K}}(z)$. Moreover, let us assume that there exists an element

$$a(z) \in \overline{\mathbb{K}}(z)(f_1(z), \ldots, f_n(z)) \cap \overline{\mathbb{K}(z)} \setminus \overline{\mathbb{K}}(z).$$

Then, by (3.5), the functions $f_1(z), \ldots, f_n(z)$ are algebraically dependent over $\overline{\mathbb{K}(z)}$ and hence over $\overline{\mathbb{K}}(z)$, which is a contradiction.

This shows in particular that when $p \mid d$, there exist regular d-Mahler extensions. In fact, in this case, there even exist regular d-Mahler extensions associated with a solution of a d-Mahler system whose coordinates are algebraically dependent over $\overline{\mathbb{K}}(z)$. Indeed, let us consider the system

$$\begin{pmatrix} f_1(z^q) \\ f_2(z^q) \\ f_3(z^q) \\ f_4(z^q) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{1}{z}\right)^q - T & -\left(\left(\frac{1}{z}\right)^q - T\right) & 0 & 0 \\ 0 & 0 & \frac{1}{1-Tz^q} & 0 \\ 0 & 0 & \frac{1}{z^q} & -\frac{1}{z^q} \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{pmatrix}. \tag{6.6}$$

Now, let us set

$$f(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\left(\left(\frac{1}{z}\right)^q - T\right)\left(\left(\frac{1}{z}\right)^{q^2} - T\right) \cdots \left(\left(\frac{1}{z}\right)^{q^n} - T\right)}$$

and

$$g(z) = \prod_{n=1}^{+\infty} \left(1 - Tz^{q^n}\right).$$

These functions are respectively $f_1(\frac{1}{z})$ and $g(z)$ introduced by L. Denis in [7,8]. According to L. Denis, $f(z)$ and $g(z)$ are analytic on $\{z \in C, |z| < \left(\frac{1}{q}\right)^{\frac{1}{q}}\}$ (even on the open unit disc for $g(z)$) and algebraically independent over $\overline{\mathbb{K}}(z)$.

We see that a solution to (6.6) is given by

$$f_1(z) = 1, f_2(z) = f(z), f_3(z) = g(z), f_4(z) = f(z)g(z).$$

Then, the Mahler extension $\overline{\mathbb{K}}(z)(f_1(z), \ldots, f_4(z))$ is regular over $\overline{\mathbb{K}}(z)$. Indeed, $f(z), g(z)$ are algebraically independent over $\overline{\mathbb{K}}(z)$, and $\overline{\mathbb{K}}(z)(1, f(z), g(z), f(z)g(z)) = \overline{\mathbb{K}}(z)(f(z), g(z))$.

6.3. **Non-regular extensions.** We have seen in the introduction of this paper that the p-Mahler extension $\mathcal{E} = \overline{\mathbb{K}}(z)(1, \sum_{n=0}^{+\infty} z^{p^n})$ is not regular over $\overline{\mathbb{K}}(z)$. In this case, $\mathcal{E}$ is an algebraic extension of $\overline{\mathbb{K}}(z)$. But there also exist non-regular transcendental q-Mahler extensions. Moreover, such an extension can be found among the simplest possible q-Mahler extensions, that is, those of the form

$$\mathcal{E} = \overline{\mathbb{K}}(z)(f(z))_{\sigma_q},$$

where $f(z)$ is a transcendental q-Mahler function.

Let us set

$$f(z) = z + \sum_{n=1}^{+\infty} F_n z^{q^n} - \sum_{n=0}^{+\infty} \frac{z^{q^n}}{1 - Tz^{q^n}},$$

where F_n is the residue modulo p of the n-th Fibonacci number (with $F_1 = 1, F_2 = 1$). By [7], $f(z)$ is a transcendental analytic function on $\{z \in C, |z| < \frac{1}{q}\}$. Moreover, we have

$$f\left(z^{q^3}\right) - 2f\left(z^q\right) + f(z) - R(z) = 0, \tag{6.7}$$

where

$$R(z) = z - z^q - z^{q^2} + z^{q^3} - \frac{z}{1-Tz} + \frac{z^q}{1-Tz^q} + \frac{z^{q^2}}{1-Tz^{q^2}}.$$

We prove the following proposition.

Proposition 6.3. *The q-Mahler extension $\mathcal{E} = \overline{\mathbb{K}}(z)(f(z))_{\sigma_q}$ is non-regular over $\overline{\mathbb{K}}(z)$.*

Proof. Let us set

$$g(z) = \sum_{n=0}^{+\infty} \frac{z^{q^n}}{1-Tz^{q^n}}$$

and

$$a(z) = z + \sum_{n=1}^{+\infty} F_n z^{q^n}.$$

First, let us notice that $g(z^q) = g(z) - \frac{z}{1-Tz}$, $a(z^{q^2}) = -a(z^q) + a(z) + z^{q^2} - z$, and $a(z)^{q^2} = -a(z)^q + a(z) + z^{q^2} - z$. On the other hand, the sequence $(a_k)_k \in \mathbb{F}_p^{\mathbb{N}}$ defined by

$$a_1 = 1 \quad \text{and} \quad \text{for } k \geq 2 \quad \text{by } a_k = \begin{cases} F_n & \text{if } k = q^n, \\ 0 & \text{otherwise} \end{cases}$$

admits unbounded gaps of consecutive zeros. Hence, $(a_k)_k$ is not eventually periodic if there exists an infinite set of non-zero Fibonacci numbers modulo p. This latter property follows from the formula $F_N^2 - F_{N-1}F_{N+1} = (-1)^{N+1} \neq 0$. We deduce that

$$a(z) \in \overline{\mathbb{K}(z)} \setminus \overline{\mathbb{K}}(z).$$

Now, we compute

$$\begin{aligned} f(z) - f(z^q) &= a(z) - g(z) - a(z^q) + g(z^q) \\ &= a(z) - a(z^q) - g(z) + g(z) - \frac{z}{1-Tz} \\ &= a(z) - a(z^q) - \frac{z}{1-Tz} \\ &= a(z^{q^2}) - z^{q^2} + z - \frac{z}{1-Tz}. \qquad (6.8) \end{aligned}$$

But $a(z^{q^2}) \in \overline{\mathbb{K}(z)} \setminus \overline{\mathbb{K}}(z)$ (if not, apply suitable Cartier operators and get $a(z) \in \overline{\mathbb{K}}(z)$, which is a contradiction). This implies that $f(z) - f(z^q) \in \mathcal{E} \cap \overline{\mathbb{K}(z)} \setminus \overline{\mathbb{K}}(z)$ and proves that $\mathcal{E}$ is not regular over $\overline{\mathbb{K}}(z)$. Proposition 6.3 is proved. □

In addition, in this case, a direct and elementary approach shows that the conclusion of Theorem 1.3 is not satisfied. First, let us prove the following lemma.

Lemma 6.4. *The functions $1, f(z), f(z^q), f\left(z^{q^2}\right)$ are linearly independent over $\overline{\mathbb{K}}(z)$. In other words, inhomogeneous equation (6.7) is minimal.*

Proof. Let us assume by contradiction that there exist polynomials

$$P_{-1}(z), \ldots, P_2(z) \in \overline{\mathbb{K}}[z],$$

not all zero, such that

$$P_{-1}(z) + P_0(z)f(z) + P_1(z)f\left(z^q\right) + P_2(z)f\left(z^{q^2}\right) = 0.$$

Then, after computations, we get

$$\begin{aligned} g(z)(P_0(z) + P_1(z) + P_2(z)) = P_{-1}(z) &+ P_1(z)P(z) + P_2(z)Q(z) + P_2(z)P\left(z^q\right) \\ &+ P_2(z)P(z) + a(z)(P_0(z) + P_2(z)) \\ &+ a\left(z^q\right)\left(P_1(z) - P_2(z)\right), \end{aligned} \tag{6.9}$$

where $P(z) = \frac{z}{1-Tz}$ and $Q(z) = z^{q^2} - z$. But $g(z)$ is transcendental over $\overline{\mathbb{K}}(z)$ [7], whereas the right-hand side of (6.9) is algebraic over $\overline{\mathbb{K}}(z)$. Hence

$$P_0(z) + P_1(z) + P_2(z) = 0, \tag{6.10}$$

and

$$\begin{aligned} P_{-1}(z) + P_1(z)P(z) + P_2(z)Q(z) + P_2(z)P\left(z^q\right) + P_2(z)P(z) + a(z)(P_0(z) + P_2(z)) \\ + a\left(z^q\right)\left(P_1(z) - P_2(z)\right) = 0. \end{aligned} \tag{6.11}$$

Now, let us notice that the function $a(z)$ seems similar to the one defined by (6.2). But in (6.2), $a(z)$ is d-Mahler with $p \nmid d$ and is transcendental over $\overline{\mathbb{K}}(z)$, whereas here $a(z)$ is q-Mahler, with $q = p^r$, and is algebraic over $\overline{\mathbb{K}}(z)$. Nevertheless, arguing as in Lemma 6.2, we get that $1, a(z), a(z^q)$ are also linearly independent over $\overline{\mathbb{K}}(z)$. Hence, by (6.11),

$$P_1(z) = P_2(z) = -P_0(z).$$

By (6.10), we get $P_0(z) = P_1(z) = P_2(z) = 0$. Finally, by (6.11), $P_{-1}(z) = 0$, which is a contradiction. Lemma 6.4 is proved. $\square$

Finally, let $\alpha \in \overline{\mathbb{K}}$, $0 < |\alpha| < 1$ be a regular number for the system associated with (6.7), that is, $\alpha \notin \left\{ \left(\frac{1}{T}\right)^{1/q^k} \right\}_{k \geq 0}$. By (6.8), we see that $f(\alpha) - f(\alpha^q) \in \overline{\mathbb{K}}$; that is, $1, f(\alpha), f(\alpha^q)$ are linearly dependent over $\overline{\mathbb{K}}$. Hence, $1, f(\alpha), f(\alpha^q), f\left(\alpha^{q^2}\right)$ are linearly dependent over $\overline{\mathbb{K}}$. But it follows from Lemma 6.4 that the conclusion of Theorem 1.3 is not satisfied.

Acknowledgments

The author would like to thank Patrice Philippon for pointing out that (2.3) follows from the work of W. Krull and for all the interesting discussions they had, which go beyond the framework of this paper, his availability, and his kindness; Julien Roques for his review of a preliminary version of the proof of Proposition 4.1; Jean-Paul Allouche for his interest and comments; the reviewers for their remarks and comments, in particular for pointing out the geometric interpretation of the notion of regular extensions; and, obviously, Boris Adamczewski for his multiple reviews and corrections of this paper and his useful advice. Any critical remarks must be exclusively addressed to the author of this paper.

References

[1] Boris Adamczewski and Colin Faverjon, *Méthode de Mahler, transcendance et relations linéaires: aspects effectifs* (French, with English and French summaries), J. Théor. Nombres Bordeaux **30** (2018), no. 2, 557–573. MR3891327

[2] Boris Adamczewski and Colin Faverjon, *Méthode de Mahler: relations linéaires, transcendance et applications aux nombres automatiques* (French, with English and French summaries), Proc. Lond. Math. Soc. (3) **115** (2017), no. 1, 55–90, DOI 10.1112/plms.12038. MR3669933

[3] Boris Adamczewski and Colin Faverjon, *Mahler's method in several variables I. The theory of regular singular systems*, preprint, 2018, 65 pp.

[4] Boris Adamczewski and Tanguy Rivoal, *Exceptional values of E-functions at algebraic points*, Bull. Lond. Math. Soc. **50** (2018), no. 4, 697–708, DOI 10.1112/blms.12168. MR3870952

[5] Yves André, *Solution algebras of differential equations and quasi-homogeneous varieties: a new differential Galois correspondence* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 2, 449–467, DOI 10.24033/asens.2218. MR3215927

[6] F. Beukers, *A refined version of the Siegel-Shidlovskii theorem*, Ann. of Math. (2) **163** (2006), no. 1, 369–379, DOI 10.4007/annals.2006.163.369. MR2195138

[7] Laurent Denis, *Indépendance algébrique des dérivées d'une période du module de Carlitz* (French, with English summary), J. Austral. Math. Soc. Ser. A **69** (2000), no. 1, 8–18. MR1767389

[8] Laurent Denis, *Indépendance algébrique de logarithmes en caractéristique p* (French, with English summary), Bull. Austral. Math. Soc. **74** (2006), no. 3, 461–470, DOI 10.1017/S0004972700040508. MR2273754

[9] David Eisenbud, *Commutative algebra*: *With a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR1322960

[10] Gwladys Fernandes, *Méthode de Mahler en caractéristique non nulle: un analogue du théorème de Ku. Nishioka* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **68** (2018), no. 6, 2553–2580. MR3897974

[11] Stéphane Fischler and Tanguy Rivoal, *Arithmetic theory of E-operators* (English, with English and French summaries), J. Éc. polytech. Math. **3** (2016), 31–65, DOI 10.5802/jep.28. MR3477864

[12] Alexander Graham, *Kronecker products and matrix calculus: with applications*, Ellis Horwood Series in Mathematics and its Applications, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1981. MR640865

[13] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157

[14] Wolfgang Krull, *Parameterspezialisierung in Polynomringen* (German), Arch. Math. **1** (1948), 56–64, DOI 10.1007/BF02038208. MR0026640

[15] Wolfgang Krull, *Parameterspezialisierung in Polynomringen. II. Das Grundpolynom* (German), Arch. Math. (Basel) **1** (1948), 129–137, DOI 10.1007/BF02039524. MR0032587

[16] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR1878556

[17] Saunders Mac Lane, *Modular fields. I. Separating transcendence bases*, Duke Math. J. **5** (1939), no. 2, 372–393, DOI 10.1215/S0012-7094-39-00532-6. MR1546131

[18] Yu. V. Nesterenko and A. B. Shidlovskiĭ, *On the linear independence of values of E-functions* (Russian, with Russian summary), Mat. Sb. **187** (1996), no. 8, 93–108, DOI 10.1070/SM1996v187n08ABEH000152; English transl., Sb. Math. **187** (1996), no. 8, 1197–1211. MR1418342

[19] Kumiko Nishioka, *New approach in Mahler's method*, J. Reine Angew. Math. **407** (1990), 202–219, DOI 10.1515/crll.1990.407.202. MR1048535

[20] Kumiko Nishioka, *Mahler functions and transcendence*, Lecture Notes in Mathematics, vol. 1631, Springer-Verlag, Berlin, 1996. MR1439966

[21] Patrice Philippon, *Groupes de Galois et nombres automatiques* (French, with English and French summaries), J. Lond. Math. Soc. (2) **92** (2015), no. 3, 596–614, DOI 10.1112/jlms/jdv056. MR3431652

[22] Bernard Randé, *Equations fonctionnelles de Mahler et applications aux suites p-régulières*, Université Bordeaux 1, 1992.

[23] Julien Roques, *On the reduction modulo p of Mahler equations*, Tohoku Math. J. (2) **69** (2017), no. 1, 55–65, DOI 10.2748/tmj/1493172128. MR3640014

[24] Hermann Ludwig Schmid, *Über die Automorphismen eines algebraischen Funktionenkörpers von Primzahlcharakteristik* (German), J. Reine Angew. Math. **179** (1938), 5–15, DOI 10.1515/crll.1938.179.5. MR1581581

[25] Jean-Pierre Serre, *Corps locaux* (French), Deuxième édition; Publications de l'Université de Nancago, No. VIII, Hermann, Paris, 1968. MR0354618

[26] Andrei Borisovich Shidlovskii, *Transcendental numbers*, translated from the Russian by Neal Koblitz, with a foreword by W. Dale Brownawell, De Gruyter Studies in Mathematics, vol. 12, Walter de Gruyter & Co., Berlin, 1989. MR1033015

[27] Joseph H. Silverman, *The arithmetic of elliptic curves*, 2nd ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR2514094

[28] Balwant Singh, *On the group of automorphisms of function field of genus at least two*, J. Pure Appl. Algebra **4** (1974), 205–229, DOI 10.1016/0022-4049(74)90022-X. MR0360600

[29] Oscar Zariski and Pierre Samuel, *Commutative algebra. Vol. II*, The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960. MR0120249

Université Claude Bernard Lyon 1, Institut Camille Jordan, 43, Boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France

Email address: `fernandes@math.univ-lyon1.fr`